Constant Elasticity of Substitution/Transformation Functions

cgemod

Contents

1. Introduction

Constant Elasticity of Substitution (CES) functions were introduced in a classic paper by Arrow *et. al.*, (1961). This functional form had many of the properties of the CD function but relaxed one of the CD function's most restrictive properties; namely the fixed elasticity of substitution of one. It is the functional form of choice in CGE models for the modelling of trade and production and is used in some models as utility functions. A major benefit of CES functions is the properties of regular nested systems of CES production function (Perroni and Rutherford, 1995); it has been shown that nested CES functions can closely approximate more flexible functional forms, such as the translog, without the need for additional restrictions and they are more computational tractable.

The short notes in this document are a refresher in CES functions; those experiencing any difficulties should refer to the standard microeconomics, mathematics for economics and mathematical economics textbooks.

2. Properties of CES/CET Functions

The CES function has the general form of

$$Y_a = \alpha_a \left[\sum_f \delta_{f,a} X_{f,a}^{-\rho_a} \right]^{-1/\rho_a} \tag{2.1}$$

where Y is the output, $X_{f,a}$ the input/factor of type f used by activity a, $\delta_{f,a}$ the share parameter for input f, and α is the technology parameter. CGE models use both the general form and the two-argument form, which can be written, when aggregating imports and domestic production, as

$$Q = a\left[\delta M^{-\rho} + (1-\delta)D^{-\rho}\right]^{-1/\rho} \tag{2.2}$$

where Q is output, D is a domestic commodity and M an imported commodity.

Implicitly the general form (2.1) is defined as a production function and the two-argument form is defined as an aggregator for domestic and imported commodities to model trade, i.e., following the Armington insight.

The CES functional from satisfies all the standard properties of functional forms used in modern economics; this is demonstrated for the two-argument form, which is adequate to demonstrate the key properties of the function and can be easily extended to the case of n arguments. Different approaches are used for the two-argument and n-argument forms simply to illustrate that different methods can be used.

Two-Argument CES/CET Functions

Homogeneity

The CES function is homogenous of degree one. Formally

$$\begin{aligned} Q &= a\left[\delta (kM)^{-\rho} + (1-\delta)(kD)^{-\rho}\right]^{-1/\rho} \\ &= a\left\{\left(k^{-\rho}\right)\left[\delta M^{-\rho} + (1-\delta)D^{-\rho}\right]\right\}^{-1/\rho}. \\ &= \left(k^{-\rho}\right)^{-1/\rho} Q = kQ \end{aligned} \tag{2.3}$$

therefore, the function is homogenous of degree one, which means it displays constant returns to scale, qualifies for application of Euler's theorem and the marginal and average products are positive and homogenous of degree zero in prices.

Marginal Products

The change in output resulting from a change in input, *ceteris paribus*, is the marginal product, i.e.,

$$\begin{aligned} MP_D = \frac{\partial Q}{\partial D} &= a\left(-\frac{1}{\rho}\right)\left[\delta M^{-\rho} + (1-\delta)D^{-\rho}\right]^{-\left(1/\rho\right)-1}(1-\delta)D^{-\rho-1} \\ &= (1-\delta)\left\{a\left[\delta M^{-\rho} + (1-\delta)D^{-\rho}\right]\right\}^{-\left((1+\rho)/\rho\right)} D^{-(1+\rho)} \\ &= \frac{(1-\delta)}{a^{\rho}}\left[\frac{a\left[\delta M^{-\rho} + (1-\delta)D^{-\rho}\right]^{-\left(1/\rho\right)}}{D}\right]^{1+\rho} \\ &= \frac{(1-\delta)}{a^{\rho}}\left[\frac{Q}{D}\right]^{1+\rho} > 0 \end{aligned} \tag{2.4}$$

and is the slope of the function, which is positive. The same expression for M is symmetrical.

Slope of Isoquants/Indifference Curves

When M is on the vertical axis and D is on the horizontal axis, the slope of the curve is

$$\frac{dM}{dD} = -\frac{MP_D}{MP_M} = -\frac{\dfrac{(1-\delta)}{a^{\rho}}\left[\dfrac{Q}{D}\right]^{1+\rho}}{\dfrac{(\delta)}{a^{\rho}}\left[\dfrac{Q}{M}\right]^{1+\rho}} = -\frac{(1-\delta)}{\delta}\left(\frac{M}{D}\right)^{1+\rho} < 0 \tag{2.5}$$

We can note that in a least cost equilibrium the slope of the relative price line will equal the slope of the curve, hence

$$-\frac{P_D}{P_M} = -\frac{(1-\delta)}{\delta}\left(\frac{M}{D}\right)^{1+\rho} \tag{2.6}$$

which resolving the minus signs and solving for the optimal ratio of M to D yields

$$\left(\frac{M^*}{D^*}\right) = \left(\frac{\delta}{(1-\delta)}\right)^{1/(1+\rho)}\left(\frac{P_D}{P_M}\right)^{1/(1+\rho)} = \left[\left(\frac{\delta}{(1-\delta)}\right)\left(\frac{P_D}{P_M}\right)\right]^{1/(1+\rho)} \tag{2.7}$$

where the asterisks indicate optimal quantities.

Elasticity of Substitution

The elasticity of substitution, σ (sigma) is defined as the percentage change in the M-D ratio divided by the percentage change in the relative prices. This is a pure number since both numerators and denominators are measured in the same units.

For the CES function the elasticity of substitution can be derived by taking logarithms of both sides of the first order condition (2.7)

$$\ln\left(\frac{M^*}{D^*}\right) = \ln\left(\frac{\delta}{(1-\delta)}\right) + \left(\frac{1}{(1+\rho}\right)\ln\left(\frac{P_D}{P_M}\right). \tag{2.8}$$

Defining the formula for the elasticity as

$$\varepsilon = \frac{d\left(\ln M^* / D^*\right)}{d\left(\ln P_D / P_M\right)} = \left(\frac{1}{(1+\rho}\right) \tag{2.9}$$

which demonstrates that ε is a constant that depends on the value of the parameter ρ.

Specifically

$$\begin{array}{lll} -1 < \rho < 0 & \Rightarrow & \varepsilon > 1 \\ \rho = 0 & \Rightarrow & \varepsilon = 1 \\ 0 < \rho < \infty & \Rightarrow & \varepsilon < 1 \end{array} \tag{2.10}$$

which is an important result. If $\varepsilon = 1$ then the function is Cobb-Douglas, which suggests that the CD function is a special case of the CES function, but as is evident if $\rho = 0$ then the exponent on the function goes to zero and the function is undefined because of division by zero. However, there is a proof that relies on *L'Hôpital*'s rule that demonstrates that as ρ tends to zero, so the function tends to CD.

For a GAMS based programme this has important implications when parameterising/calibrating CES/CET functions (see below).

Convex and Concave Versions

This is one of the very rare occasions when the second derivative is useful for the standard functional forms used in economics.[1] The distinction between a CES and CET function depends upon the sign on the exponent. Consider the second derivatives of two CES functions. For the first where the arguments are M and D, as has been the case until now, if (1 + ρ) is > 0 then

$$\frac{d^2M}{dD^2} = \frac{(1-\delta)}{\delta}(1+\rho)M^{1+\rho}D^{-\rho-2} > 0 \tag{2.11}$$

and the function is convex, i.e., a CES function, e.g., an indifference curve or an isoquant.

But where the arguments are E and D and if (1 + ρ) is < 0 then

$$\frac{d^2E}{dD^2} = \frac{(1-\gamma)}{\gamma}.(1+\rho)E^{1+\rho}D^{-\rho-2} \quad < 0 \tag{2.12}$$

and the function is concave, i.e., a CET function, e.g., a production possibility frontier or a consumption possibility frontier.

This will turn out to be important when modelling trade and multi-product activities.

<u>*n*-Argument CES/CET Functions</u>

For the n-argument form of the CES function an approach used is to define a profit function and from that derive the expressions for factor demands consistent with profit maximisation from the first order conditions. Given the n-argument form

$$Y_a = \alpha_a\left[\sum_f \delta_{f,a} X_{f,a}^{-\rho_a}\right]^{-1/\rho_a}$$

The profit equation can be written as

$$\begin{aligned}\pi_a &= P_a.Y_a - \sum_f W_{f,a}.X_{f,a} \\ &= P_a.\alpha_a\left[\sum_f \delta_f.X_{f,a}^{-\rho_a}\right]^{-1/\rho_a} - \sum_f W_{f,a}.X_{f,a}\end{aligned} \tag{2.13}$$

[1] Typically, the functional forms used by economists are such that the second order conditions are such that the first order conditions are always consistent with an optimal solution given the specification of the functional form.

where P_a is the output price and for activity a and $W_{f,a}$ is the price for factor f in activity a. Differentiating the profit equation with respect to aa arbitrary factor f and setting the differential equal to zero produces

$$\frac{\partial \pi_a}{\partial X_{f,a}} = P_a.\alpha_a.\left[\sum_f \delta_{f,a}.X_{f,a}^{-\rho_a}\right]^{-1/\rho_a - 1}.\delta_{f,a}X_{f,a}^{-\rho_a - 1} - W_{f,a} = 0 \tag{2.14}$$

which can then be solved for the factor price as

$$\begin{aligned} W_{f,a} &= P_a.\alpha_a.\left[\sum_f \delta_{f,a}.X_{f,a}^{-\rho_a}\right]^{-\left(\frac{1+\rho_a}{\rho_a}\right)}.\delta_{f,a}.X_{f,a}^{-(\rho_a+1)} \\ &= P_a.Y_a\left[\sum_f \delta_{f,a}.X_{f,a}^{-\rho_a}\right]^{-1}.\delta_{f,a}.X_{f,a}^{-(\rho_a+1)} \end{aligned} \qquad (2.15)^2$$

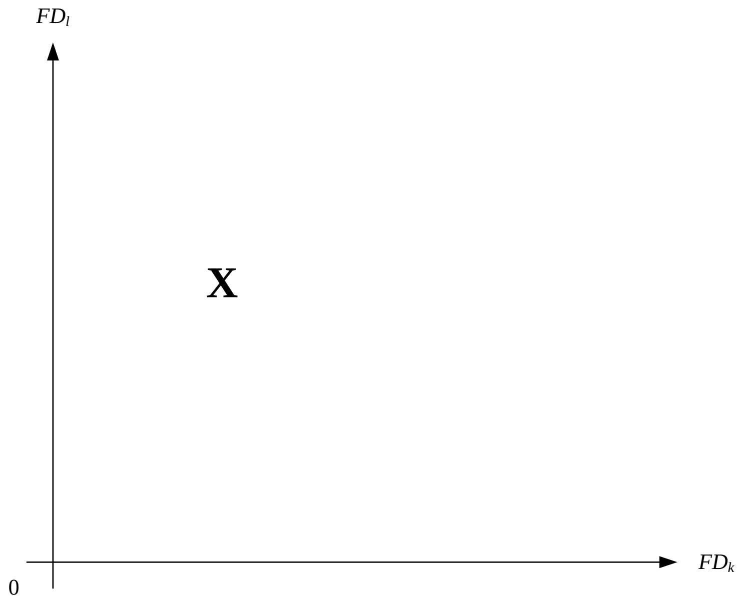
This produces a standard first-order condition expression for factor demands consistent with profit maximisation. This method avoids the need for an explicit zero-profit condition in a CGE model by embedding the zero-profit condition in the FOCs for optimal factor allocation.

[2] Note how the second line of (2.15) is derived by factoring the expression is square brackets in the first line using the formula for Y_a in (2.1) and the law of exponents.

3. Calibrating CES/CET Functions in CGE Models

The problem of calibration in CGE models is that the observed data only identify transactions values, whereas the models need to identify separately the prices and quantities of inputs and outputs and the parameters of the chosen functional forms. For the case of two inputs and one output the problem can be summarised with a simple diagram (Figure 1), where **X** marks the observed transaction values of the two inputs. The problem thus reduces to identifying the prices of the two inputs, the price and quantity of the output and the parameters of the production function.

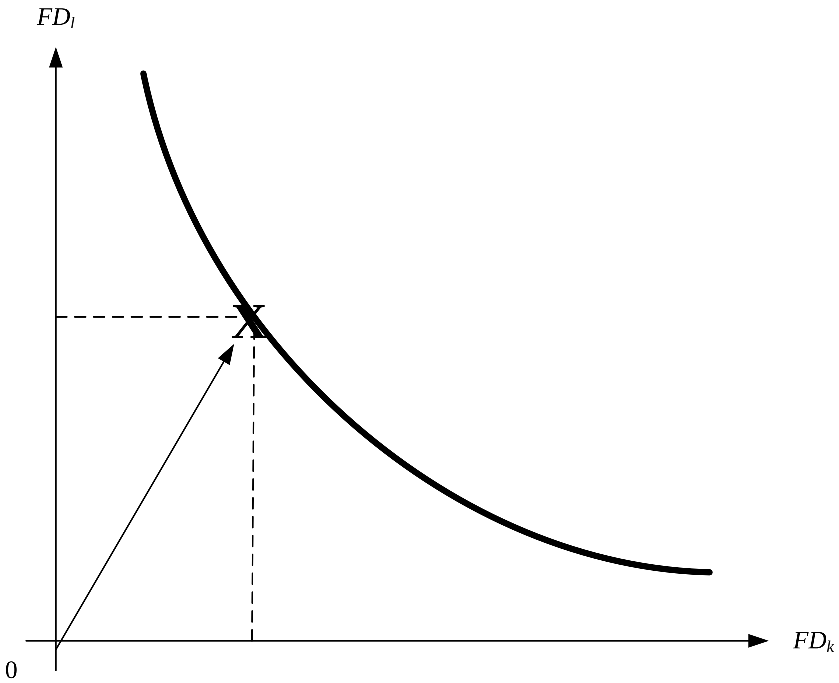
Figure 1 Calibration Problem for CGE Models



Since the functional forms used are all linear homogeneous the model is defined in terms of relative prices. Thus, if as is often the case, we do not have data for prices, we can assign arbitrary prices for the inputs and outputs. Typically, the prices will be set equal to one so the quantity units are ‘value’ quantities. Thus, the problem is further reduced to the identification of the parameters of the production function: namely the elasticity of substitution, the shift or efficiency parameter and the share parameters.

Given the transactions data for the two inputs, the functional form (CES), the elasticity of substitution and the assumption of technical and economic efficiency, the point X in the diagram must lie on the isoquant that represents the known level of the output and the curvature of the isoquant, at the point X, is known from the known functional form and the elasticity of substitution.

Figure 2 Calibration in CGE Models



The calibration process works backwards. The modeler must tell the programme that the functional form is CES or CET, the transactions values, the prices and the elasticity of substitution.

1. The functional forms are defined by the model;
2. the transactions values are recorded in the SAM;
3. the prices are either exogenously imposed from real quantities or a price normalisation rule is used to produce value quantities; and
4. the elasticities are provided exogenously, i.e., as part of the database

Given this information, and the appropriate equations, the model can determine the quantities of output/input, and the share and shift parameters for the function. Unlike the CD function the necessity of providing the model with a value for every CES/CET parameter highlights the issue of the elasticities but does give the modeler flexibility.

Two Argument Calibration

The example of calibration given here is that used for the 123-model used in Module O3 of the Practical CGE modelling course. The 123-model only has a single commodity and therefore the variables are not defined for sets.

CES Function

Defining the elasticity of substitution for a CES function as σ, then if the elasticity value is given exogenously the elasticity parameter is

$$\rho = \frac{1}{\sigma} - 1 \tag{2.16}$$

Assuming the transaction data report an equilibrium state, then the 1st order condition in (2.7) holds for the prices and quantities for the recorded transactions, i.e.,

$$\left(\frac{M0^*}{D0^*}\right) = \left[\left(\frac{\delta}{(1-\delta)}\right)\left(\frac{P0_D}{P0_M}\right)\right]^{1/(1+\rho)} \tag{2.17}$$

where the zeros represent initial values. Solving (2.17) for δ produces

$$\delta = \frac{P0_M * M0^{(\rho+1)}}{P0_M * M0^{(\rho+1)} + P0_D * D0^{(\rho+1)}} \tag{2.18}$$

which since the initial quantities and prices can be derived from the transaction values and the price normalisation rules means that δ can be computed because all the RHS terms are known. Given the initial quantities and prices, ρ and δ the function can be solved for the technology parameter a, i.e.,

$$\alpha = \frac{Q0}{\left[\delta M0^{-\rho} + (1-\delta) D0^{-\rho}\right]^{-1/\rho}} \tag{2.19}$$

This demonstrates how the CES function is calibrated by working back from the functional form, transaction values and the elasticity parameter when assuming the transaction values represent an equilibrium. In GAMS, the code is

```
rhoc      = (1/sigma) - 1 ;
delta      = (PM0*QM0**(rhoc+1))
                 /(PM0*QM0**(rhoc+1) + PDD0*QDD0**(rhoc+1)) ;;
```

```
ac        = Q0/(delta*QM0**(-rhoc) +(1-delta)*QD0**(-rhoc))**(-1/rhoc) ;
```

CET Function

Defining the elasticity of substitution for a CET function as *Ω*, then, if the elasticity value is given exogenously, the elasticity parameter is

$$\rho = \frac{1}{\Omega} + 1 \tag{2.20}$$

Assuming the transaction data report an equilibrium state then the 1st order condition in (2.7), substituting appropriately, holds for the prices and quantities for the recorded transactions, i.e.,

$$\frac{E0}{D0} = \left[\left(\frac{\gamma}{1-\gamma} \right) \left(\frac{PD0}{PE0} \right) \right]^{\left(1/_{1+\rho} \right)} \tag{2.21}$$

where the zeros represent initial values. Solving (2.21) for γ produces

$$\gamma = \frac{1}{\left(D0/_{E0} \right)^{(1+\rho_T)} \left(P0_D/_{P0_E} \right) + 1} \tag{2.22}$$

which since the initial quantities and prices can be derived from the transaction values and the price normalisation rules means that γ can be computed because all the RHS terms are known. Given the initial quantities and prices, ρ and γ the function can be solved for the technology parameter *a*, i.e.,

$$a = \frac{Q0}{\left[\delta E0^{-\rho} + (1-\delta) D0^{-\rho} \right]^{1/\rho}} \tag{2.23}$$

NB: the different sign on the function.

This demonstrates how the CET function is calibrated by working back from the functional for, transaction values and the elasticity parameter when assuming the transaction values represent an equilibrium. In GAMS, the code is[3]

```
rhot        = (1/omega) + 1 ;

gamma        = 1/(1+PDS0/PE0*(QE0/QDS0)**(rhot-1));

at           = QX0/(gamma*QE0**rhot + (1-gamma)*QDS0**rhot)**(1/rhot) ;
```

3 NB: certain $ control terms have not been reported to make the formulae less opaque.

<u>*n*-Argument Calibration</u>

CES Function

The example of calibration given here is that used for the value-added production function in *smod_t2* model used in Module O6 of the Single Country/Global CGE modelling courses. The *smod_t2* model has multiple activities (*a*) and factors (*f*). The data used for the *smod_t2* model does not include factor use taxes that may be paid by activities that are indexed on *a* and *f*; this simplifies the exposition. A more general version is used in the STAGE_t and ANARRES_t models of the Single Country and /Global CGE modelling courses.

Defining the elasticity of substitution for a CES function as σ, then if the elasticity values are given exogenously the elasticity parameters (with the super script *va* to indicate value added and the subscript *a* to indicate activity *a*) are

$$\rho_a^{va} = \frac{1}{\sigma_a^{va}} - 1 \tag{2.24}$$

Assuming the transaction data report an equilibrium state then the 1st order condition holds for the prices and quantities for the recorded transactions, is

$$WFA0_{f,a} = PVA0_a * QVA0_a \left(\sum \delta_{f,a}^{va} * FD0_{f,a}^{\left(-\rho_a^{va}\right)} \right)^{-1} * \delta_{f,a}^{va} * FD0_{f,a}^{\left(-\rho_a^{va}-1\right)} \tag{2.25}$$

where $WFA_{f,a}$ is the price of factor *f* in activity *a*, PVA_a and QVA_a are the price and quantity of value added in activity *a*, $\delta_{f,a}^{va}$ is the share parameter of factor *f* in activity *a*, and $FD_{f,a}$ is the quantity of factor *f* used/demanded in activity *a*. The zeros represent initial values. Solving (2.25) for δ produces

$$\delta_{f,a}^{va} = \frac{WFA0_{f,a} * FD0_{f,a}^{\left(1+\rho_a^{va}\right)}}{\sum_f WFA0_{f,a} * FD0_{f,a}^{\left(1+\rho_a^{va}\right)}} \tag{2.26}$$

which since the initial quantities and prices can be derived from the transaction values and the price normalisation rules means that $\delta_{f,a}^{va}$ can be computed because all the RHS terms are known. Given the initial quantities and prices, ρ and δ the function can be solved for the technology parameter $adva_a$, i.e.,

$$adva_a = \frac{QVA0_a}{\left(\sum_f \delta_{f,a}^{va} * FD0_{f,a}^{\left(-\rho_a^{va}\right)}\right)^{\left(-1/\rho_a^{va}\right)}} \tag{2.27}$$

This demonstrates how the CES function is calibrated by working back from the functional for, transaction values and the elasticity parameter when assuming the transaction values represent an equilibrium. In GAMS, the code is[4]

```
rhova(a)        = (1/ELASTX(a,"sigmava")) - 1 ;

deltava(f,a)    = (WFA(f,a)*(FD0(f,a))**(1+rhova(a)))

                   /SUM(fp,WFA0(fp,a)*(FD0(fp,a)) **(1+rhova(a))) ;

adva0(a)        = QVA0(a)/(SUM(f,deltava(f,a)*FD0(f,a)

                  **(-rhova(a))))**(-1/rhova(a));
```

CET Function

The example of calibration given here is that used for the export supply functions in the ANARRES model used in the Global CGE modelling course. The ANARRES model has multiple commodities (*c*) exported to multiple destinations (*w*) by multiple sources (*r*). For ease of exposition the export taxes are ignored in the illustration below.

The elasticity of transformation for a CET function, *Ω*, is exogenously assigned and the elasticity parameter value for exports of commodity c by region r is

$$\rho_{c,r} = \frac{1}{\Omega_{c,r}} + 1 \tag{2.28}$$

Assuming the transaction data report an equilibrium state then the 1st order condition holds for the prices and quantities for the recorded transactions, so that the quantity of exports of *c* to region *w* from region *r* (*QER*) is

$$QER_{c,w,r} = QE_{c,r} * \left(\frac{PER_{c,w,r}}{PE_{c,r} * \gamma_{c,w,r} * atr_{c,r}^{\rho_{c,r}^{e}}}\right)^{\left(1/\rho_{c,r}^{e} - 1\right)} \tag{2.29}$$

4 NB: certain $ control terms have not been reported to make the formulae less opaque.

where *QE* is the total volume of exports of *c* from region *r* (*QER*), *PER* is the price exports of *c* to region *w* from region *r*, γ is the share parameter for exports of *c* to region *w* from region *r* and *atr* is the shift parameter of exports of *c* from region *r*.

The share parameter can then be calibrated as value shares modified by the elasticity parameters

$$\gamma_{c,w,r} = \left(\frac{PER0_{c,w,r} * QER0_{c,w,r}^{\left(1-\rho^e_{c,r}\right)}}{\sum_w PER0_{c,w,r} * QER0_{c,w,r}^{\left(1-\rho^e_{c,r}\right)}} \right) \tag{2.30}$$

where the zeros represent initial values. Note how given the elasticity parameter and since the initial quantities and prices can be derived from the transaction values and the price normalisation rules means that γ can be computed because all the RHS terms are known. Given the initial quantities and prices, ρ and γ the function can be solved for the technology parameter *atr*, i.e.,

$$atr_{c,r} = \frac{QE0_{c,r}}{\left[\sum \gamma_{c,w,r} * QER0_{c,w,r}^{\rho^e_{c,r}} \right]^{1/\rho^e_{c,r}}} \tag{2.23}$$

NB: the different sign on the function.

This demonstrates how the CET function is calibrated by working back from the functional form, transaction values and the elasticity parameter when assuming the transaction values represent an equilibrium. In GAMS, the code is[5]

```
rhoe(c,r)         = ((1/mod_elastre(c,r)) + 1) ;
gammar(c,w,r)     = (PER0(c,w,r)*QER0(c,w,r)**(1-rhoe(c,r)) ) /
                        SUM(wp(c,wp,r), PER0(c,wp,r)*QER0(c,wp,r)
                           **(1-rhoe(c,r)) );
atr(c,r)          = QE0(c,r)/SUM(w, gammar(c,w,r)*QER0(c,w,r)
                              **(rhoe(c,r)) )**(1/(rhoe(c,r))) ;
```

[5] NB: certain $ control terms have not been reported to make the formulae less opaque.

References

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