Constant Elasticity of Substitution/Transformation Functions

1

Outline

- Introduction
- General Properties of CES/CET functions
  - Homogeneity
    - Euler's theorem
  - MP and elasticity
- Substitution v Transformation Functions
  - 2nd order conditions
  - Convexity v Concavity
- Calibrating CES/CET functions

2

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Introduction

- Purposes
  - Review
    - Makes sure we are all on the ‘same page’
  - Concepts
  - Properties
  - Product exhaustion
  - Calibration
- Use
  - CES – production, utility, import demand
  - CET – export supply, multi-product activities, factor ‘transformation’
  - Nested functions

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CES/CET Properties

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2-Argument Form: Homogeneity

Formula $Q = a\left[\delta M^{-\rho} + (1-\delta)D^{-\rho}\right]^{-1/\rho}$

$$Q = a\left[\delta\left(kM\right)^{-\rho} + (1-\delta)\left(kD\right)^{-\rho}\right]^{-1/\rho}$$
$$= a\left\{\left(k^{-\rho}\right)\left[\delta M^{-\rho} + (1-\delta)D^{-\rho}\right]\right\}^{-1/\rho}$$
$$= \left(k^{-\rho}\right)^{-1/\rho} Q = kQ$$

Homogeneous degree one

CRTS
Euler's theorem
MP & AP positive
Homogenous of degree zero in prices.

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5

2-Argument Form: Marginal Product

$$MP_D = \frac{\partial Q}{\partial D} = a\left(-\frac{1}{\rho}\right)\left[\delta M^{-\rho} + (1-\delta)D^{-\rho}\right]^{-\left(1/\rho\right)-1}\left(1-\delta\right)D^{-\rho-1}$$
$$= \left(1-\delta\right)\left\{a\left[\delta M^{-\rho} + (1-\delta)D^{-\rho}\right]\right\}^{-\left(\left(1+\rho\right)/\rho\right)} D^{-(1+\rho)}$$
$$= \frac{\left(1-\delta\right)}{a^{\rho}}\left[\frac{a\left[\delta M^{-\rho} + (1-\delta)D^{-\rho}\right]^{-\left(1/\rho\right)}}{D}\right]^{1+\rho}$$
$$= \frac{\left(1-\delta\right)}{a^{\rho}}\left[\frac{Q}{D}\right]^{1+\rho} > 0$$

Marginal Products

Laws of exponents

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2-Argument Form: First Order Condition

$$\frac{dM}{dD} = -\frac{MP_D}{MP_M} = -\frac{\frac{(1-\delta)}{a^\rho}\left[\frac{Q}{D}\right]^{1+\rho}}{\frac{(\delta)}{a^\rho}\left[\frac{Q}{M}\right]^{1+\rho}} = -\frac{(1-\delta)}{\delta}\left(\frac{M}{D}\right)^{1+\rho} < 0$$

Slope of curve

$$-\frac{P_D}{P_M}$$

Slope of budget constraint

$$-\frac{P_D}{P_M} = -\frac{(1-\delta)}{\delta}\left(\frac{M}{D}\right)^{1+\rho}$$

1st Order condition

$$\left(\frac{M^*}{D^*}\right) = \left(\frac{\delta}{(1-\delta)}\right)^{1/(1+\rho)}\left(\frac{P_D}{P_M}\right)^{1/(1+\rho)} = \left\{\left(\frac{\delta}{(1-\delta)}\right)\left(\frac{P_D}{P_M}\right)\right\}^{1/(1+\rho)}$$

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7

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2-Argument Form: Substitution Elasticity

$$\ln\left(\frac{M^*}{D^*}\right) = \left(\frac{1}{(1+\rho)}\right)\left(\ln\left(\frac{\delta}{(1-\delta)}\right) + \ln\left(\frac{P_D}{P_M}\right)\right)$$

Logs of 1st Order Condition

$$\varepsilon = \frac{d\left(\ln M^* / D^*\right)}{d\left(\ln P_D / P_M\right)} = \left(\frac{1}{(1+\rho)}\right)$$

Elasticity of Sub^n

$$\begin{aligned} -1 < \rho < 0 \quad &\Rightarrow \quad \varepsilon > 1 \\ \rho = 0 \quad &\Rightarrow \quad \varepsilon = 1 \\ 0 < \rho < \infty \quad &\Rightarrow \quad \varepsilon < 1 \end{aligned}$$

Properties

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2-Argument Form: Second Order Conditions

Arguments: *M* and *D*, AND iff (1 + ρ) is > 0

$$\frac{d^2M}{dD^2} = \frac{(1-\delta)}{\delta}(1+\rho)M^{1+\rho}D^{-\rho-2} > 0$$

CES
Isoquants & Indifference curves

Arguments: *M* and *D*, AND iff (1 + ρ) is < 0

$$\frac{d^2M}{dD^2} = \frac{(1-\delta)}{\delta}.(1+\rho)M^{1+\rho}D^{-\rho-2} < 0$$

CET
ppf & cpf

For CET functions: use γ to distinguish from CES functions

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9

n-Argument Form

Formula

$$Y_a = \alpha_a \left[\sum_f \delta_{f,a} X_{f,a}^{-\rho_a} \right]^{-1/\rho_a}$$

Profit Equation

$$\pi_a = P_a.Y_a - \sum_f W_{f,a}.X_{f,a}$$

$$= P_a.\alpha_a \left[\sum_f \delta_{f,a}.X_{f,a}^{-\rho_a} \right]^{-1/\rho_a} - \sum_f W_{f,a}.X_{f,a}$$

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10

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n-Argument Form: First Order Condition

Profit Equation: Partial Differentiation *wrt* $X_{f,a}$

$$\frac{\partial \pi_a}{\partial X_{f,a}} = P_a.\alpha_a.\left[\sum_f \delta_{f,a}.X_{f,a}^{-\rho_a}\right]^{-1/\rho_a - 1}.\delta_{f,a}X_{f,a}^{-\rho_a - 1} - W_{f,a} = 0$$

First Order Condition for Factor Demands

$$W_{f,a} = P_a.\alpha_a.\left[\sum_f \delta_{f,a}.X_{f,a}^{-\rho_a}\right]^{-\left(\frac{1+\rho_a}{\rho_a}\right)}.\delta_{f,a}.X_{f,a}^{-(\rho_a+1)}$$

$$= P_a.Y_a\left[\sum_f \delta_{f,a}.X_{f,a}^{-\rho_a}\right]^{-1}.\delta_{f,a}.X_{f,a}^{-(\rho_a+1)}$$

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11

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Calibration

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12

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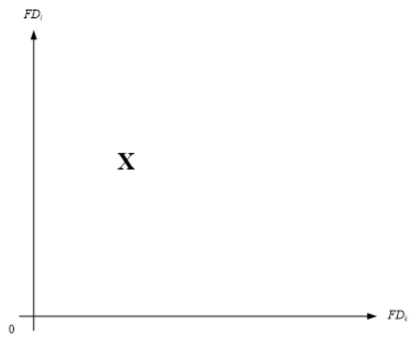
Calibration



Observed transactions data

The problem is the identification of the parameters of the function: namely the elasticity of substitution, the shift or efficiency parameter and the share parameters

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13

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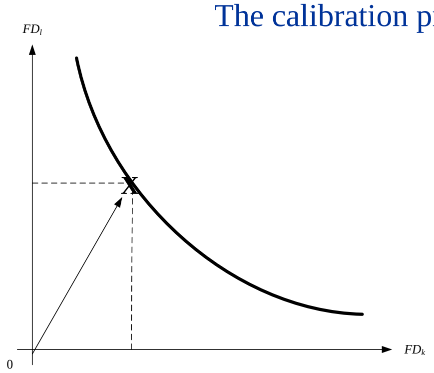
Calibration

The calibration process works backwards



Info provided to the model:
the functional form, e.g., CD, CES, etc., the transaction values, the prices and the elasticity of substitution.

Assumption: transaction values are an equilibrium

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14

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2-Argument Form: CES Calibration

1st Order condition $\left(\frac{M0}{D0}\right)=\left(\left(\frac{\delta}{(1-\delta)}\right)\left(\frac{P0_D}{P0_M}\right)\right)^{1/(1+\rho)}$

$$\rho=\frac{1}{\sigma}-1$$ Elasticity parameter

$$\delta=\frac{P0_M * M0^{(\rho+1)}}{P0_M * M0^{(\rho+1)}+P0_D * D0^{(\rho+1)}}$$ Share parameter

$$a=\frac{Q0}{\left[\delta M0^{-\rho}+(1-\delta)D0^{-\rho}\right]^{-1/\rho}}$$ Shift parameter

In GAMS

```
rhoc      = (1/sigma) - 1 ;
delta      = (PM0*QM0**(rhoc+1))
              /(PM0*QM0**(rhoc+1) + PDD0*QDD0**(rhoc+1)) ;
ac        = Q0/(delta*QM0**(-rhoc) +(1-delta)*QD0**(-rhoc))
                 **(-1/rhoc) ;
```

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15

2-Argument Form: CET Calibration

1st Order condition $\frac{E0}{D0}=\left(\left(\frac{\gamma}{1-\gamma}\right)\left(\frac{P0_D}{P0_E}\right)\right)^{\left(1/1-\rho\right)}$

$$\rho=\frac{1}{\Omega}+1$$ Elasticity parameter

$$\gamma=\frac{1}{\left(D0/E0\right)^{(1-\rho)}\left(P0_D/P0_E\right)+1}$$ Share parameter

$$a=\frac{Q0}{\left[\gamma E0^{\rho}+(1-\gamma)D0^{\rho}\right]^{1/\rho}}$$ Shift parameter

In GAMS

```
rhot         = (1/omega) + 1 ;
gamma          = 1/(1+PDS0/PE0*(QE0/QDS0)**(rhot-1));
at             = QX0/(gamma*QE0**rhot + (1-gamma)*QDS0**rhot)
                          **(1/rhot) ;
```

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16

n-Argument: CES Calibration

1st Order condition $WFA0_{f,a} = PVA0_a * QVA0_a \left(\sum \delta^{va}_{f,a} * FD0_{f,a}^{\left(-\rho^{va}_a\right)} \right)^{-1} * \delta^{va}_{f,a} * FD0_{f,a}^{\left(-\rho^{va}_a - 1\right)}$

$$\rho^{va}_a = \frac{1}{\sigma^{va}_a} - 1$$

Elasticity parameter

$$\delta^{va}_{f,a} = \frac{WFA0_{f,a} * FD0_{f,a}^{\left(1+\rho^{va}_a\right)}}{\sum_f WFA0_{f,a} * FD0_{f,a}^{\left(1+\rho^{va}_a\right)}}$$

Share parameter

$$adva_a = \frac{QVA0_a}{\left(\sum_f \delta^{va}_{f,a} * FD0_{f,a}^{\left(-\rho^{va}_a\right)} \right)^{\left(-1/\rho^{va}_a\right)}}$$

Shift parameter

```
rhova(a)      = (1/ELASTX(a,"sigmava")) - 1 ;
deltava(f,a)  = (WFA(f,a)*(FD0(f,a))**(1+rhova(a)))
                  /SUM(fp,WFA0(fp,a)*(FD0(fp,a)) **(1+rhova(a)));
adva0(a)      = QVA0(a)/(SUM(f,deltava(f,a)*FD0(f,a)
                 **(-rhova(a))))**(-1/rhova(a));
```

In GAMS

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17

n-Argument: CET Calibration

1st Order condition $QER0_{c,w,r} = QE0_{c,r} * \left(\frac{PER0_{c,w,r}}{PE0_{c,r} * \gamma_{c,w,r} * atr_{c,r}^{\rho^e_{c,r}}} \right)^{\left(1/\rho^e_{c,r} - 1\right)}$

$$\rho_{c,r} = \frac{1}{\Omega_{c,r}} + 1$$

Elasticity parameter

$$\gamma_{c,w,r} = \left(\frac{PER0_{c,w,r} * QER0_{c,w,r}^{\left(1-\rho^e_{c,r}\right)}}{\sum_w PER0_{c,w,r} * QER0_{c,w,r}^{\left(1-\rho^e_{c,r}\right)}} \right)$$

Share parameter

$$atr_{c,r} = \frac{QE0_{c,r}}{\left[\sum \gamma_{c,w,r} * QER0_{c,w,r}^{\rho^e_{c,r}} \right]^{1/\rho^e_{c,r}}}$$

Shift parameter

```
rhoe(c,r)        = ((1/mod_elastre(c,r)) + 1) ;
gammar(c,w,r)    = (PER0(c,w,r)*QER0(c,w,r)**(1-rhoe(c,r)) ) /
                      SUM(wp(c,wp,r), PER0(c,wp,r)*QER0(c,wp,r)
                         **(1-rhoe(c,r)) );
 atr(c,r)        = QE0(c,r)/SUM(w, gammar(c,w,r)*QER0(c,w,r)
                             **(rhoe(c,r)) )**(1/(rhoe(c,r))) ;
```

In GAMS

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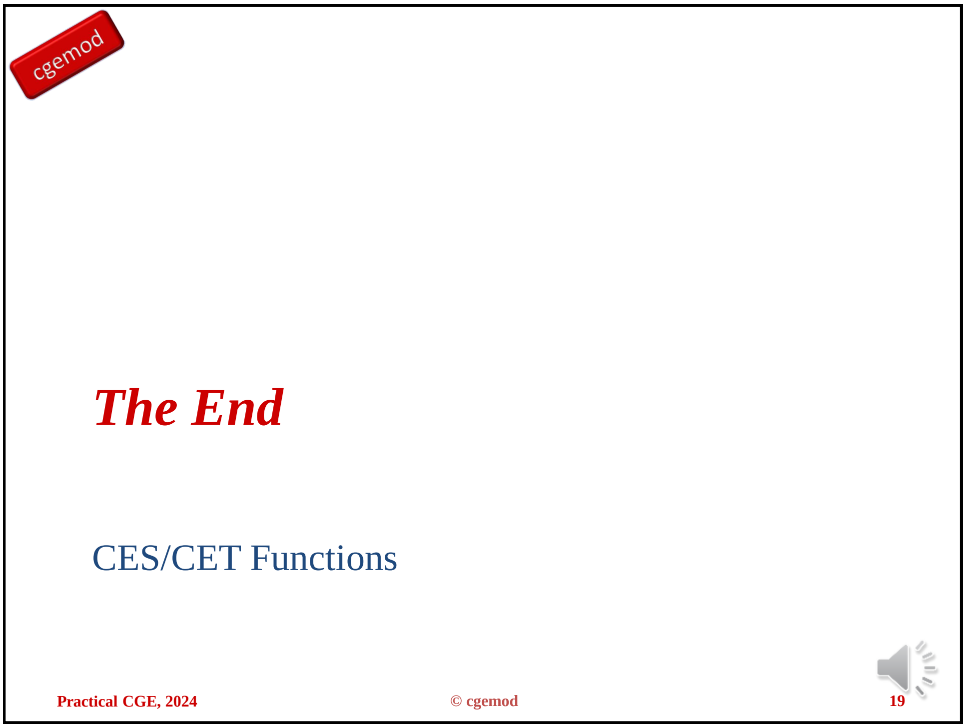
18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19

19