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Cobb Douglas Functions

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Outline

- Introduction
- CD Production function
  – MP, MRTS, elasticity, returns to scale
  – Long run cost minimisation
- CD Utility function
  – MU, MRS, budget shares
- Product exhaustion
  – Euler's theorem
- Calibrating a CD function

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Introduction

- Purposes
  - Review
    - Makes sure we are all on the same page
  - Concepts
  - Properties of linear homogenous functions
  - Product exhaustion
  - Calibration
- Use
  - CD production functions
  - CD utility functions
  - Using a 'simple' function allows us to concentrate on techniques and concepts

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Theory (mathematics)

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General Form in Production

Production Technology

$$X = \alpha . \prod_f F_f^{\beta_f} \qquad X = \alpha . L^{\beta_l} K^{\beta_k}$$

Marginal Products

$$MP_L = \frac{\partial X}{\partial L} = \beta_l \alpha L^{\beta_l - 1} K^{\beta_k} = \beta_l \left(\alpha L^{\beta_l} K^{\beta_k} \right) L^{-1}$$

$$= \beta_l \frac{X}{L} = \beta_l \left(AP_L \right)$$

$$MP_K = \beta_k \frac{X}{K} = \beta_k \left(AP_k \right)$$

$MRTS_{L,K}$

$$MRTS_{L,K} = \frac{\partial X / \partial L}{\partial X / \partial K} = \frac{\beta_l . X / L}{\beta_k . X / K} = \frac{\beta_l}{\beta_k} . \frac{K}{L}$$

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Long Run Cost Minimisation

Problem

Min $\quad C = wL + rK$

sto $\quad \bar{X} = \alpha L^{\beta_l} K^{\beta_k}$

Lagrangian

$$\phi = w.L + r.K + \lambda \left(X - \alpha L^{\beta_l} K^{\beta_k} \right)$$

Partial derivatives

$$\frac{\partial \phi}{\partial L} = w - \lambda \beta_l \alpha L^{\beta_l - 1} K^{\beta_k} = w - \lambda \beta_l . \frac{X}{L} = 0$$

$$\frac{\partial \phi}{\partial K} = r - \lambda \beta_k \alpha L^{\beta_l} K^{\beta_k - 1} = r - \lambda \beta_k . \frac{X}{K} = 0$$

$$\frac{\partial \phi}{\partial \lambda} = X - \alpha L^{\beta_l} K^{\beta_k} = 0$$

1st Order Condition

$$\frac{w}{\beta_l . \frac{X}{L}} = \lambda = \frac{r}{\beta_k . \frac{X}{K}} \Rightarrow \frac{w}{r} = \frac{\beta_l . \frac{X}{L}}{\beta_k . \frac{X}{K}} = \frac{\beta_l}{\beta_k} . \frac{K}{L}$$

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Elasticity of Substitution

$$\sigma = \frac{\%\Delta \text{ in } \left(K/L\right)}{\%\Delta \text{ in } \left(w/r\right)} \quad \text{and} \quad w/r = MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{\beta_l \left(X/L\right)}{\beta_k \left(X/K\right)} = \frac{\beta_l K}{\beta_k L}$$

$$\sigma = \frac{d\left(K/L\right)\cdot\left(L/K\right)}{d\left(\frac{\beta_l K}{\beta_k L}\right)\cdot\left(\frac{\beta_k L}{\beta_l K}\right)}$$

Elasticity of Sub^{n}

$$\sigma = \frac{\left(\frac{\beta_l}{\beta_k}\right).d\left(K/L\right)}{\left(\frac{\beta_l}{\beta_k}\right)\cdot d\left(K/L\right)} = 1$$

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Returns to Scale

If a function is homogeneous, the degree of homogeneity is a measure of returns to scale

$$X_0 = \alpha L^{\beta_l} K^{\beta_k}$$

Initial state

$$X^* = \alpha\left(kL\right)^{\beta_l}\left(kK\right)^{\beta_k}$$
$$= \left(\alpha.L^{\beta_l}K^{\beta_k}\right)k^{(\beta_l+\beta_k)}$$
$$= k^{(\beta_l+\beta_k)}X_0$$

Increase *L* and *K* by *k*

$$\beta_l + \beta_k < 1 \Rightarrow DRTS$$
$$\beta_l + \beta_k = 1 \Rightarrow CRTS$$
$$\beta_l + \beta_k > 1 \Rightarrow IRTS$$

Degree of homogeneity

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General Form in Consumption

$$U = \phi . \prod_c X_c^{\gamma_c} = \phi \cdot X_1^{\gamma_1} \cdot X_2^{\gamma_2}$$ **Utility function**

$$MU_1 = \frac{\partial U}{\partial X_1} = \phi \cdot \gamma_1 \cdot X_1^{\gamma_1 - 1} \cdot X_2^{\gamma_2} = \frac{\gamma_1 U}{X_1}$$ **Marginal Utility**

$$\frac{MU_1}{MU_2} = \frac{\left(\frac{\gamma_1 U}{X_1}\right)}{\left(\frac{\gamma_2 U}{X_2}\right)} = \frac{P_1}{P_2}$$ **1st Order Condition**

$$\frac{X_2}{X_1} = \frac{P_1}{P_2} \cdot \left(\frac{\gamma_2}{\gamma_1}\right)$$ **Rearranged**

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Demand Equations

$$Y = P_1 \cdot X_1 + P_2 \cdot X_2$$ **Budget Constraint**

$$Y = P_1 \cdot X_1 + P_2 \cdot \left(X_1 \cdot \frac{P_1}{P_2} \cdot \left(\frac{\gamma_2}{\gamma_1}\right)\right)$$ **Optimal Consumption**

$$X_1 = \frac{\gamma_1 \cdot Y}{P_1} \qquad X_2 = \frac{\gamma_2 \cdot Y}{P_2}$$ **Demand Equations**

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Product Exhaustion

$$Y = P_1 \cdot X_1 + P_2 \cdot X_2$$ **Accounting ID**

$$Y = P_1 \cdot \left(\frac{\gamma_1 \cdot Y}{P_1} \right) + P_2 \cdot \left(\frac{\gamma_2 \cdot Y}{P_2} \right)$$ **Optimal quantities**

$$Y = (\gamma_1 + \gamma_2) \cdot Y$$

$IFF \quad \gamma_1 + \gamma_2 = 1$ **Product exhaustion**

$Y = Y$

IF a Cobb-Douglas function has constant returns to scale then Euler's theorem can be applied

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Empirics (calibration)

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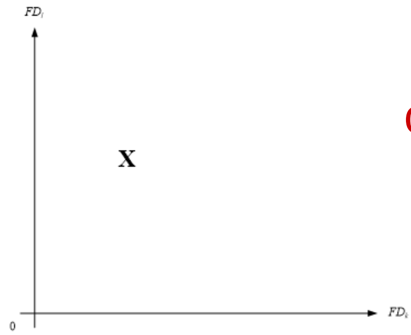
Calibration



Observed transactions data

The problem is the identification of the parameters of the function: namely the elasticity of substitution, the shift or efficiency parameter and the share parameters

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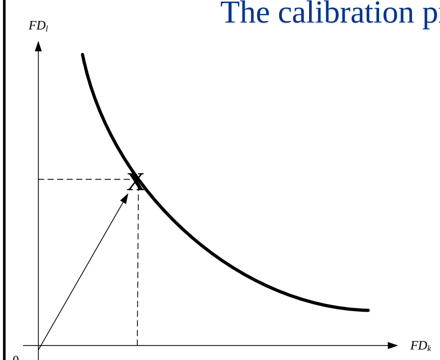
Calibration

The calibration process works backwards



Info provided to the model:
the functional form, e.g., CD, CES, etc., the transactions values, the prices and the elasticity of substitution.

```
alpha(f,a)= SAM(f,a)/SUM(fp,SAM(fp,a)) ;
ad(a)     = QX0(a)/PROD(f, FD0(f,a)**alpha(f,a)) ;
```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The End

Cobb Douglas Functions

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