General Equilibrium Theory

1. Introduction

A defining characteristic of partial equilibrium (PE) models is the presumption that events in different markets take place in isolation from events in other markets; not only in the product markets but also in the factor markets. However, it is likely that events do not in general take place in isolated markets. If we constrain the analyses to consider issues relating to the consumption of items that account for **very** small shares of expenditure, we may regard the PE approach as a reasonable approximation; but if we are considering issues where the products are ‘food’ and ‘other products’ then we might argue that we would expect the inter relationships to be sufficiently large as to justify quantifying. This is the realm of General Equilibrium (GE) analysis.

GE is the bedrock of microeconomics. While most undergraduate and postgraduate microeconomic theory texts begin with PE analyses of consumption and production all those theories have been derived, to a greater or lesser extent, from GE theory, e.g., we can justify PE theory only to the extent that we can argue that the GE effects are sufficiently small as to justify ‘discounting’. In this sense, it is regrettable that ‘modern’ microeconomic textbooks have increasingly reduced the attention to GE analyses.

A key tool for diagrammatic representation of GE theory is the Edgeworth box. We will develop this first in the context of defining ‘efficiency in production’ and thence to the definition of the ‘production possibilities frontier’, we will then extend the analysis by examining the conditions for ‘efficiency in exchange’ from which we can define a ‘utility possibility frontier’. Having derived the basic model of general equilibrium in diagram form we will, briefly, relate the results of the efficiency conditions to the efficiency conditions identified by the PE analyses and show how they depend upon the GE theory. We will conduct all this analysis in the context of a simple closed economy with 2 commodities, 2 activities, 2 factors, and 2 households, which is the conventional approach followed in undergraduate and most postgraduate textbooks; this representation can be shown to encompass all the elements of the circular flow.[1]

An advantage of developing the analyses in this way is that it forms the basis for the development of the simplest possible Computable General Equilibrium (CGE) model. This course will then proceed to convert this simple theoretical model into a simple practical model that can then be used to demonstrate GE interactions.

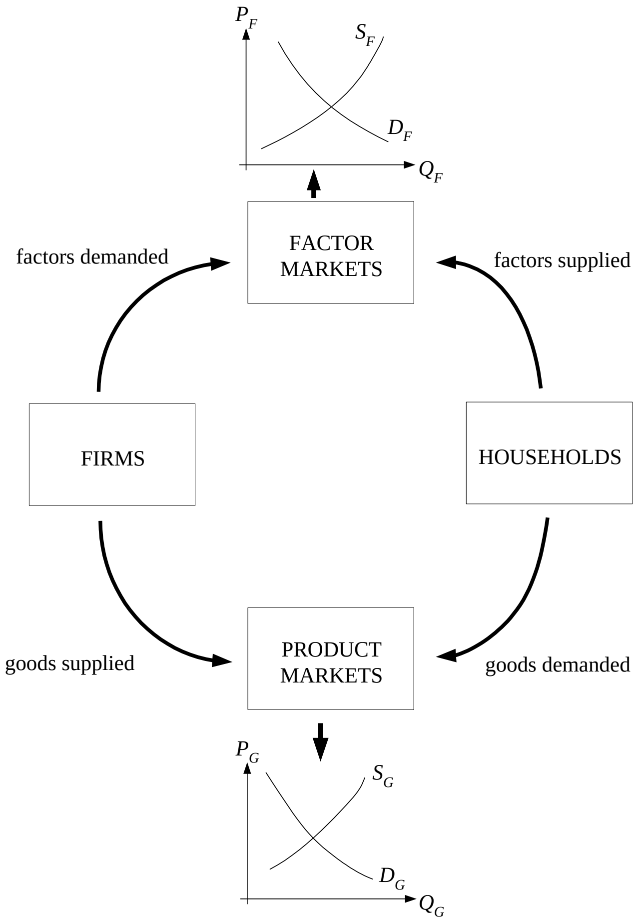
2. Circular Flow

In a closed economy, the circular flow can be represented in a very simple form (Figure 1). In product markets, commodities/products/goods are supplied by activities/industries/firms and demanded by households. In factor markets, factors are supplied by households and demanded by activities. If there is no option to save or borrow, then households, if they are rational, will spend all their incomes from factor sales making purchases on product markets. They will also make decisions about how many factors to supply to the factor markets based on their preferences over consumption and leisure and their endowments. Activities on the other hand will make decisions about the quantities of factors to employ based on the demand for products from households, the available technologies and the supply (functions) of factors. These decisions will, in theory, take place simultaneously.

Hence we have all the elements of a circular flow. What we need to do is examine how they might interact, and then how we might turn this simple system into a computable model.

[1] The conventional textbook presentation makes the (implicit) assumption that commodities and activities are identical. We relax this assumption to produce a more general set of arguments that will better carry over into the context of CGE models.

Figure 1 Closed Economy Circular Flow



3. Assumptions

We make the standard assumptions that are made with respect to the behaviour of households and activities. For households, we assume

AC1: Rationality

- the consumer seeks to maximise utility given the choices available, her tastes and preferences and the constraints under which she must operate.

AC2: Ordinal Utility

- the consumer can rank the utility provide by different consumption bundles.

AC3: Completeness

- the consumer can rank all consumption bundles.

AC4: Transitivity

If A > B and B > C, then A > C

AC5: More is preferred to less

$$\frac{\partial U}{\partial q_i} \geq 0 \quad i = 1, \cdots, n.$$

AC6: Diminishing Marginal Rate of Substitution

i.e., indifference curves are convex to the origin.

AC7: Full Information

- Full knowledge of all relevant information.

For activities, we assume

AP1: Large number of buyer and sellers.

Buyers and sellers are price takers

AP2: Homogenous product.

AP3: Free entry and exit.

An absence of rivalry/strategic behaviour

AP4: Profit maximisation.

AP5: Perfect factor mobility.

This ensures full employment of all factors in the long run

AP6: Perfect knowledge

Including technology, i.e.,

$$\frac{\partial X}{\partial L} = MP_L > 0 \text{ and } \frac{\partial^2 X}{\partial L^2} = \frac{\partial (MP_L)}{\partial L} < 0$$

Diminishing Marginal Rate of Technical Substitution

i.e., isoquants are convex to the origin.

AP7: No government intervention.

We will also make a series of general assumptions. We will restrict our analyses to a very much-simplified worldview. We will assume that we have a 2*2*2*2 model[2], partly because this allows us to work with diagrams, thus

AG1: 2 commodities – *X* and *Y*

AG2: 2 activities – each produces a single product

2 This representation separates commodities and activities – hence the extra ‘2’. This is so that this presentation is consistent with the approach needed in the more general CGE model developed later in the course.

AG3: 2 households – *A* and *B*

AG4: 2 factors of production –*L* and *K*

AG5: Each household has a fixed supply of *L* and *K*

AG5: Time horizon – long run

AG6: Perfect competition

AG7: Perfect knowledge

AG8: no government

AG9: no trade

AG10: no savings or investment.

The assumptions of no government, no savings/investment and no trade (and others), will be relaxed later, but for now are made to keep things as simple as possible

4. Efficiency in Production

The standard PE presentation of equilibrium in two product markets (*X* and *Y*) is illustrated in Figure 2, where $X_1/X_2/Y_1/Y_2$ are isoquants to produce different quantities of commodities *X* and *Y*. The illustration contains the presumption that technologies are infinitely divisibly so that infinitely many combinations of labour (*L*) and capital (*K*) can be used to produce *X* and *Y*, and that the further an isoquant is from the origin the greater the level of output. The optimal combinations of *L* and *K* used to produce each product are determined by the ratios of the exogenously imposed factor prices; these are illustrated by the 'budget' lines whose slopes are defined by the relative prices of the factors. Thus, the tangency points between the isoquants and the 'budget' lines represent the least cost of producing a given quantity of each commodity.

This is the standard presentation of efficiency in production used in undergraduate microeconomics.

In a GE context with fixed factor supplies, if a certain quantity of the two factors is used to produce commodity *X* then the quantity available to produce *Y* is defined. Hence the decisions about how much of *X* and *Y* to produce are interdependent; an Edgeworth box provides a simple and elegant way to illustrate this interdependency.

Figure 2 Equilibrium in Two Product Markets

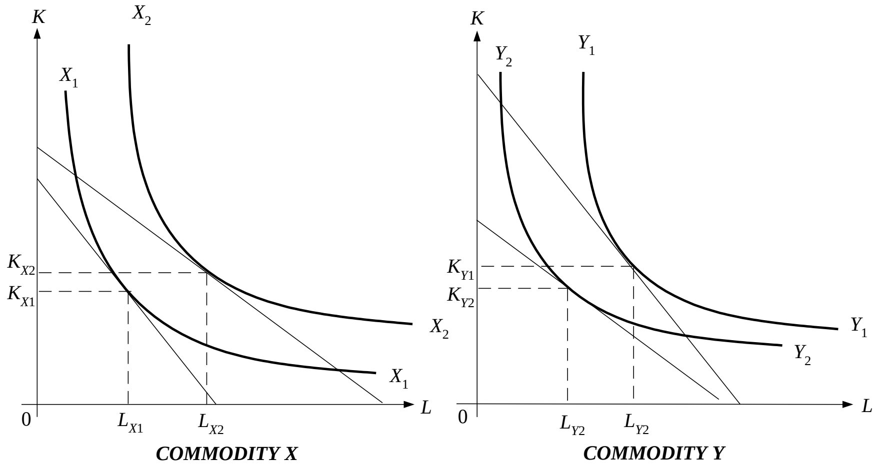
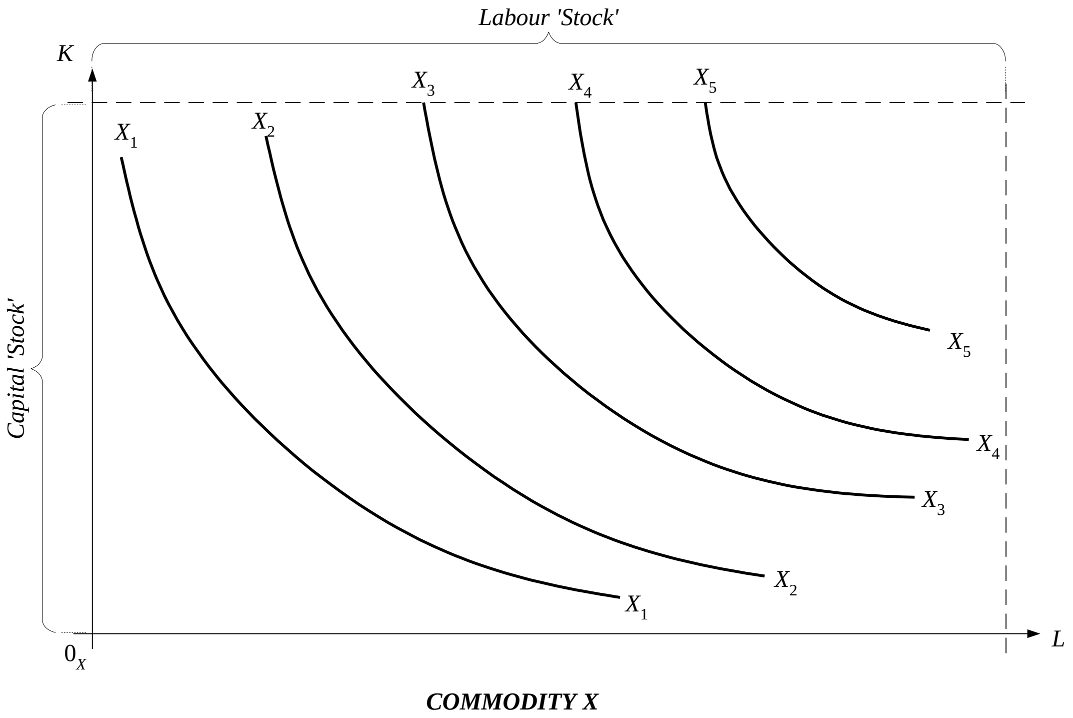


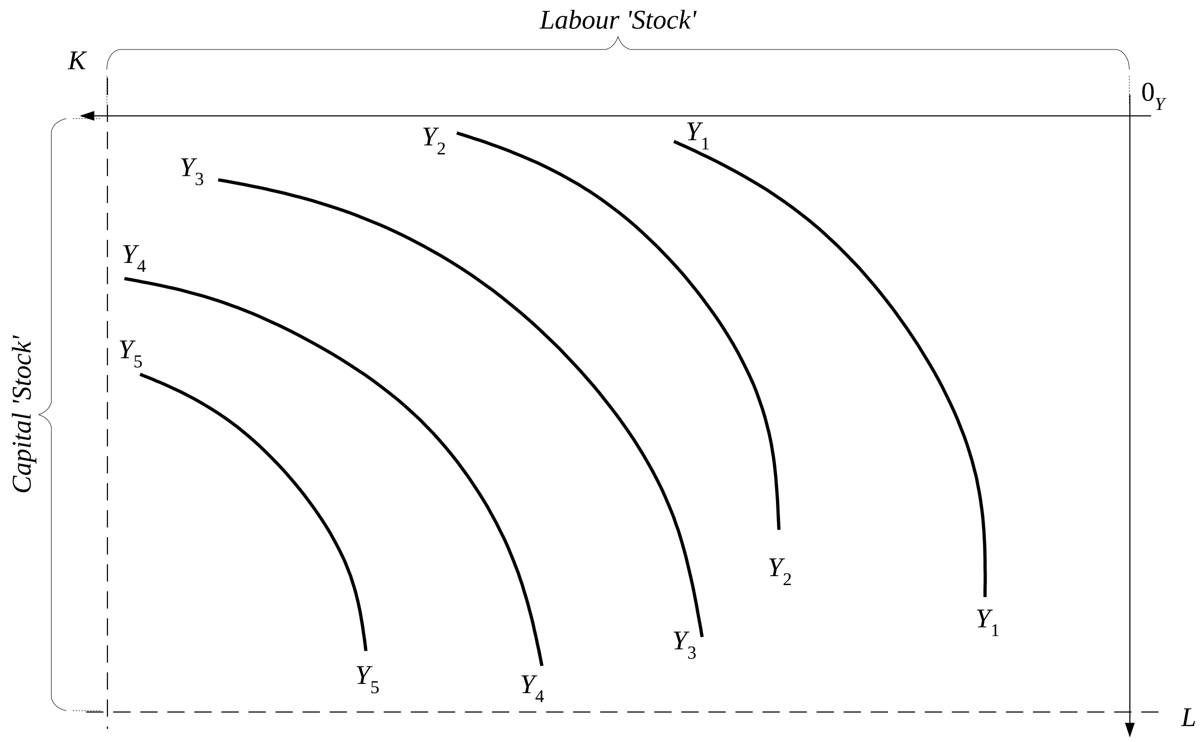
Figure 3 illustrates the constraints on the production of commodity X for given total supplies of factor *L* and *K*. The lengths of the vertical and horizontal axes are determined by the total quantities of *K* and *L*, respectively. Given the available technology for producing *X* it is possible to define a family of isoquants that identify all the possible combinations of *L* and *K* that can be used to produce all possible quantities of *X*, subject to the constraints imposed by the supply of the factors, i.e., the isoquants are everywhere dense.

Figure 3 Production of Commodity X



A similar figure can be drawn for the production of commodity *Y*; the lengths of the axes will be identical (total factor supplies are fixed) but the pattern of the isoquants will be different because the technologies to produce *X* and *Y* differ. Again, the isoquants are everywhere dense. Figure 4 illustrates the situation for commodity *Y* BUT transposed so that the origin is in the upper right-hand corner rather than the lower left-hand corner.

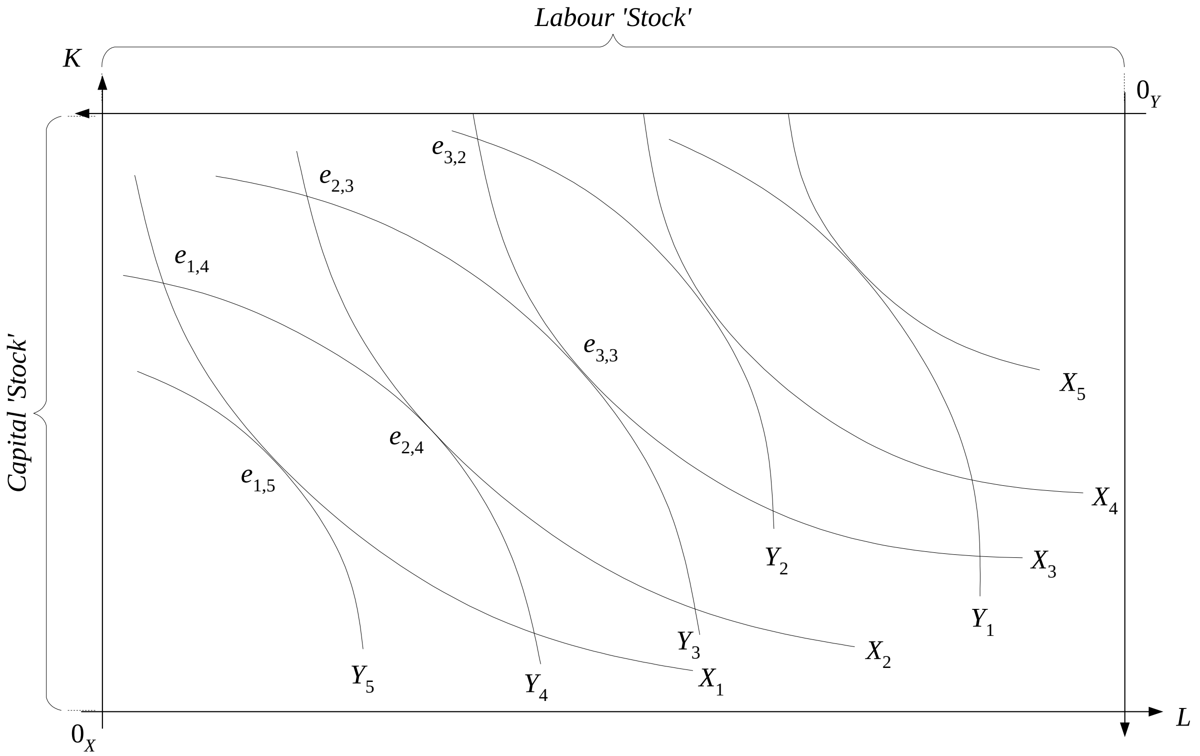
Figure 4 Production of Commodity Y



COMMODITY Y

Since the dimensions of the figures for the production of commodities *X* and *Y* are identical they can be overlaid on each other to form what is known as an Edgeworth box in production; this is illustrated in Figure 5. Given that the isoquants for *X* and *Y* are everywhere dense they will intersect at points that represent various combinations of *X* and *Y* that can be produced given the constraints of technology and factor supplies. Consider, for instance, the intersections labeled $e_{2,3}$ and $e_{3,2}$. At these intersections X_2 of *X* and Y_3 of *Y* and X_3 of *X* and Y_2 of *Y*, respectively, are being produced; these are inefficient combinations of *X* and *Y* to produce. Starting at $e_{2,3}$ on the isoquant Y_3, and keeping the quantity of *Y* produced constant, then the quantity of *X* produced can be increased by changing the combinations, and quantities, of factors used to produce Y_3. This releases quantities of *L* and *K* that can then be used to produce more of X; the quantity of X produced will increase until X_3 is being produced and the tangency labeled $e_{3,3}$ is achieved. This is illustrated as a transition along the Y_3 isoquant consistent with increases in the production of X. Beyond the tangency point the quantity of Y produced will remain constant but the quantity of X produced will decline.

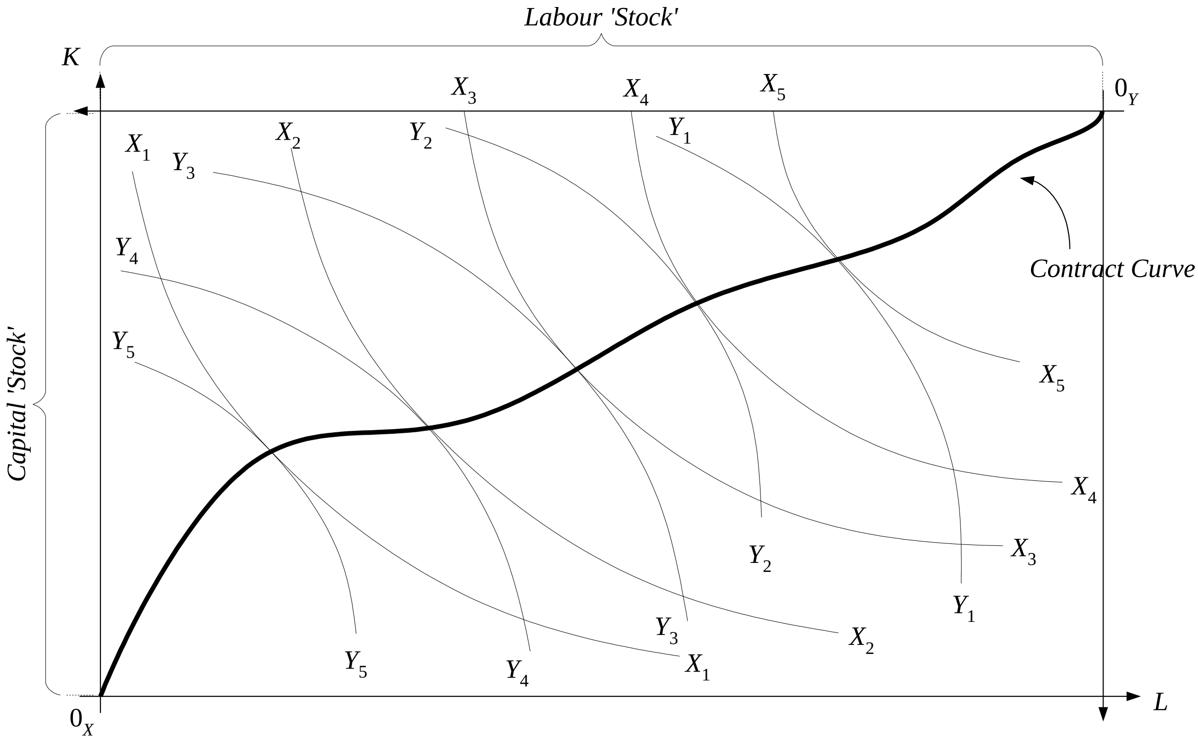
Figure 5 Efficiency and Edgeworth Box in Production



This concept of efficiency can be extended for all possible levels of production of one commodity for all possible levels of production for the other commodity. For each level of production of one commodity there is an efficient level of production of the other commodity represented by the tangency points. These tangency points can be linked together to produce a 'contract' curve that, by definition, starts at one origin[3] and ends at the other. The contract curve illustrates all combinations of the commodities X and Y that can be produced efficiently given the constraints. If the quantities of factors increase the dimensions of the Edgeworth box increase and the contract curve will change. If the technologies change the isoquants will change and the contract curve will change.

[3] At each origin, all the factors are being used to one commodity.

Figure 6 Edgeworth Box in Production and the Contract Curve



Production Efficiency Conditions

Formally, if we have two twice continuously differentiable production functions

$$X = x\left(L^X, K^X\right) \qquad \text{and} \qquad Y = y\left(L^Y, K^Y\right)$$

for which we can define an arbitrary isoquant, e.g.,

$$X^* = x\left(L^X, K^X\right)$$

along which the slope can be identified, by total differentiation. Noting that along an isoquant the change in output is zero, then

$$0 = \frac{\partial X}{\partial K}.dK^X + \frac{\partial X}{\partial L}.dL^X$$

which can be rearranged to give us the standard optimality conditions from PE producer theory

$$-\frac{dK^X}{dL^X}=\frac{\partial X/\partial L}{\partial X/\partial K}=\frac{MP_L^X}{MP_K^X}=MRTS_{KL}^X$$

which is equal to the ratio of the factor prices

$$MRTS_{KL}^X=\frac{w}{r}.$$

This is of course mirrored by the optimality conditions for the production of *Y*.

But along the contract curve the slopes of the isoquants for *X* and *Y* are identical. Hence

$$MRTS_{KL}^X=MRTS_{KL}^Y \quad \text{or} \quad \left(\frac{\partial X/\partial L}{\partial X/\partial K}\right)^X=\left(\frac{\partial Y/\partial L}{\partial Y/\partial K}\right)^Y$$

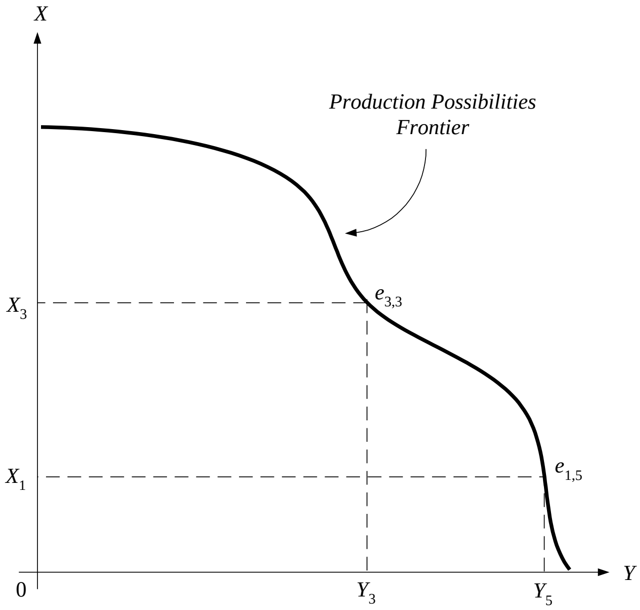
and therefore, the ratio of factor prices used in both production processes will be identical.

$$MRTS_{KL}^x=MRTS_{KL}^y=\frac{w}{r}$$

Efficiency in Product Mix

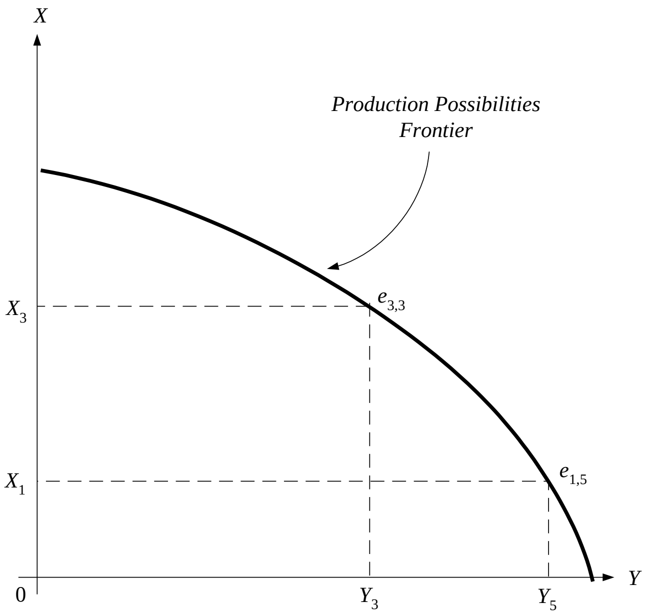
If we plot all the output combinations along the (production) contract curve generated in factor space into product space, we are identifying a production possibility frontier (Figure 7). By the definition of the contract curve, any point within the *ppf* is inefficient, since more of one product can be produced without reducing the amount of the other product produced, and any point outside the *ppf* is infeasible with the present combination of endowments and technologies.

Figure 7 Production Possibilities Frontier



In general, the *ppf* is not necessarily a smooth curve. But for pragmatic convenience, and given the standard functional forms used by economists, we will represent the *ppf* as a smooth curve.

Figure 8 Production Possibilities Frontier



Product-Mix Efficiency Conditions

Consider now the slope of the *ppf*, which we will call the marginal rate of product transformation (*MRPT*). By definition, the *MRPT* is

$$MRPT_{XY} = -\frac{dY}{dX} = \frac{MC_X}{MC_Y}$$

since it represents the rate at which one output can be transformed into another.

By definition

$$MC_X = \frac{d\left(TC_X\right)}{dX} \qquad \text{and} \qquad MC_Y = \frac{d\left(TC_Y\right)}{dY}$$

so, we can express the ratio of the *MC*s as

$$\frac{MC_X}{MC_Y} = \frac{d\left(TC_X\right)}{d\left(TC_Y\right)}.\frac{dY}{dX}.$$

But if factor prices are fixed, the change in total costs is given by the 'weighted' sum of the changes in input quantities, i.e.,

$$\begin{aligned} d\left(TC_X\right) &= w.\left(dL_X\right) + r\left(dK_X\right) \\ d\left(TC_Y\right) &= w.\left(dL_Y\right) + r\left(dK_Y\right) \end{aligned}$$

and hence the ratio of changes in total costs can be written as

$$\frac{d\left(TC_x\right)}{d\left(TC_y\right)} = \frac{w.\left(dL_x\right) + r\left(dK_x\right)}{w.\left(dL_y\right) + r\left(dK_y\right)}.$$

But we are assuming full employment and only two industries and therefore

$$dL_X = -dL_Y \qquad \text{and} \qquad dK_X = -dK_Y$$

and therefore, the ratio of total costs, in competitive equilibrium under our assumptions is

$$\frac{d\left(TC_X\right)}{d\left(TC_Y\right)} = \frac{w.\left(-dL_Y\right) + r\left(-dK_Y\right)}{w.\left(dL_Y\right) + r\left(dK_Y\right)} = -1.$$

Consequently, the ratio of *MC*s is equal to the *MRPT*, i.e.,

$$\frac{MC_X}{MC_Y} = -1.\frac{dY}{dX} = MRPT_{XY}$$

and more importantly, since

$$MC_X = P_X \quad \text{and} \quad MC_Y = P_Y$$

the MRPT is equal to the ratio of the product prices

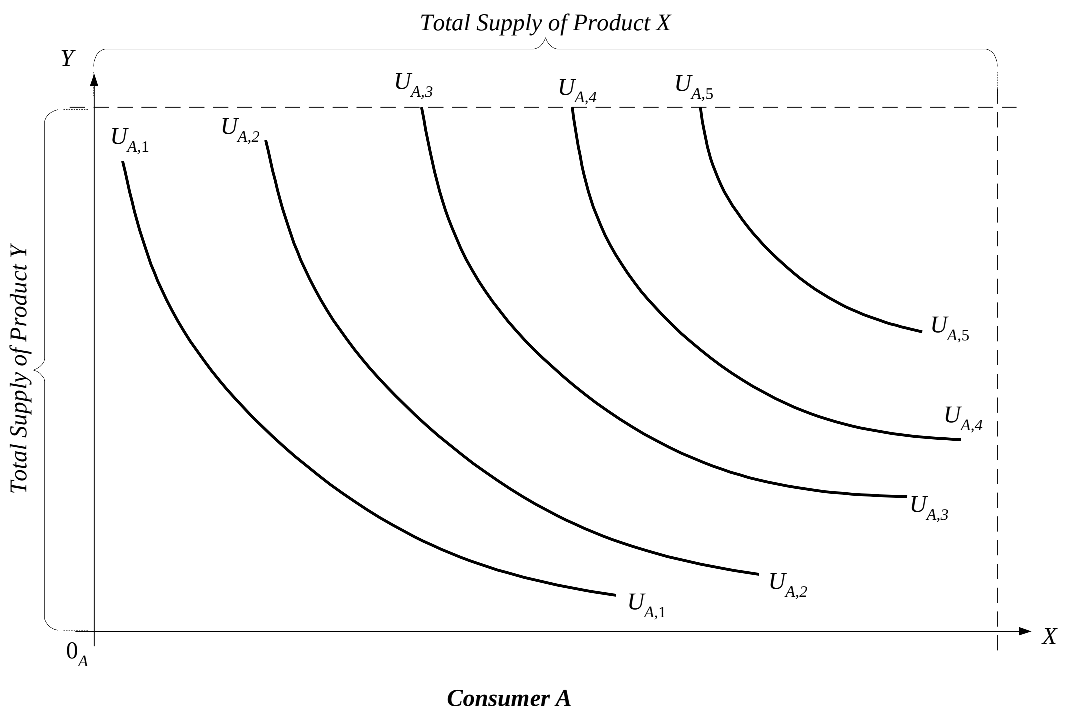
$$\frac{MC_x}{MC_y} = \frac{P_x}{P_y} = MRPT_{yx}.$$

The importance of this will become clear shortly.

5. Efficiency in Exchange

The situation in exchange can be represented is the same manner. Given technology and factor quantities it is possible to define the maximum quantities of *X* and *Y* that can be supplied efficiently, i.e., the points on the production possibility frontier. For the moment assume that a decision has been taken to produce X_2 and Y_2, i.e., at point $e_{2,2}$ on the *ppf* in Figure 8. Given this information we can draw a figure (Figure 9) with the lengths of the axes representing the total supplies of *X* and *Y*.

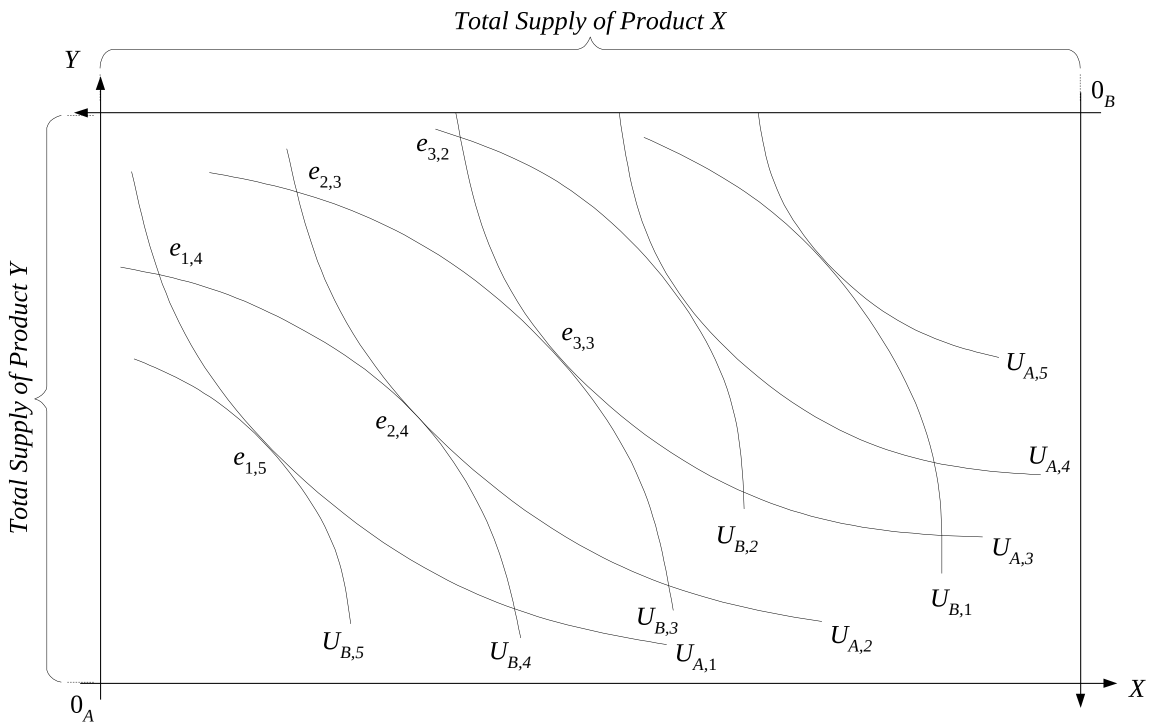
Figure 9 Consumption by A



Given the preferences of household *A* it is possible to define a family of indifference curves that identify the levels of utility from consuming all the possible combinations of *X* and *Y*, subject to the constraints imposed by the supply of the commodities, i.e., the indifference curves are everywhere dense. This illustrates all possible combinations commodities *X* and *Y* that can be consumed by household *A*; the optimal combination of *X* and *Y* consumed will depend on the relative prices of *X* and *Y* and the income available to *A*. The same can be done for household *B*.

The figures for *A* and *B* can be overlaid on each other, with the origin for *A* (0_A) in the lower let-hand corner and that for *B* (0_B) in the upper right-hand corner, to produce and Edgeworth box in consumption space (Figure 10).

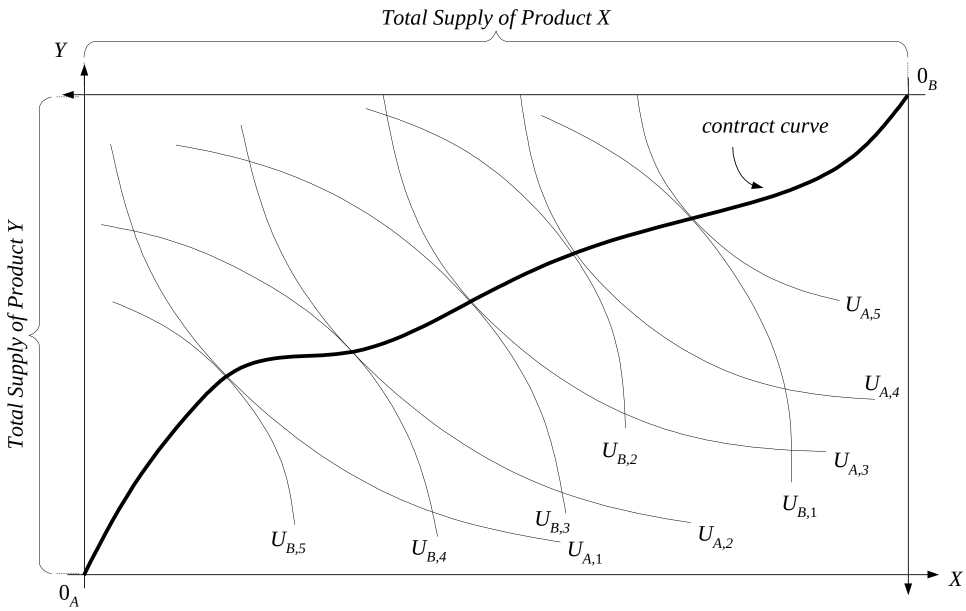
Figure 10 Efficiency and Edgeworth Box in Consumption



Given that the indifference curves for *A* and *B* are everywhere dense they will intersect at points that represent various combinations of *X* and *Y* that can be consumed given the constraints of preferences and commodity supplies. Consider, for instance, the intersections labeled $e_{2,3}$ and $e_{3,2}$. At these intersections, utilities of $U_{A,2}$ for *A* and $U_{B,3}$ for *B* and $U_{A,3}$ for *A* and $U_{B,2}$ for *B*, respectively, are being achieved; these are inefficient combinations of utility. Starting at $e_{2,3}$ on the indifference curve $U_{B,3}$, and keeping the utility of *B* constant, then the utility of *A* can be increased by changing the combinations, and quantities, of commodities consumed used to realise $U_{B,3}$. This releases quantities of *X* and *X* that can then be used to realise more utility for *A*; the quantity of utility realised will increase until $U_{A,3}$ is being realised and the tangency labeled $e_{3,3}$ is achieved. This is illustrated as a transition along the $U_{B,3}$ indifference curve consistent with increases in the utility of *A*. Beyond the tangency point the utility of *B* will remain constant but the utility of *A* will decline.

This a contract curve can be defined representing all the efficient combination of the consumption of *X* and *Y* by *A* and *B*.

Figure 11 Edgeworth Box in Consumption and the Contract Curve



The contract curve defines the loci of Pareto efficient points, where Pareto efficiency is defined as a combination of levels of utilities such that it is not possible to make one consumer better off without making at least one other worse off. In that sense a Pareto improvement is defined as movement towards a contract curve that makes one consumer better off without making at least one other consumer worse off.

<u>Consumption Efficiency Conditions</u>

Formally we have two utility functions

$$U^A = u^A\left(X^A, Y^A\right) \qquad \text{and} \qquad U^B = u^B\left(X^B, Y^B\right)$$

and for any arbitrary level of utility we can specify an indifference curve

$$U^* = u(X,Y)$$

we know, by definition, that the level of utility along the indifference curve is constant, i.e.,

$$0 = \frac{\partial U}{\partial X}.dX + \frac{\partial U}{\partial Y}.dY$$

which can be rearranged to give the standard optimality condition from PE consumer theory, i.e.,

$$-\frac{dY}{dX} = \frac{\partial U / \partial X}{\partial U / \partial Y} = \frac{MU_X}{MU_Y} = MRS_{XY}.$$

Furthermore, as shown before the MRS is equal to

$$MRS_{XY} = \frac{P_Y}{P_X}.$$

But along the contract curve the slopes of the indifference curves for *A* and *B* are identical. Hence

$$MRS^A_{XY} = MRS^B_{XY}$$
$$\left(\frac{\partial U / \partial X}{\partial U / \partial Y}\right)^A = \left(\frac{\partial U / \partial X}{\partial U / \partial Y}\right)^B$$

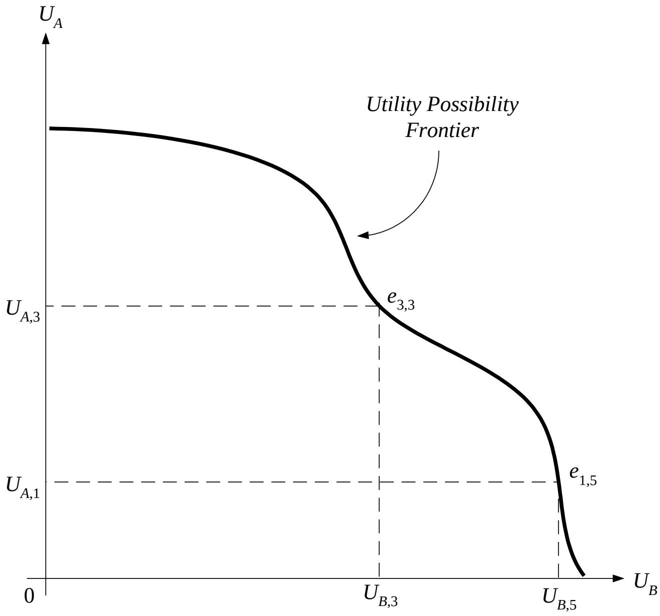
and therefore the prices faced by each consumer at the optima are identical, i.e.,

$$MRS^A_{XY} = MRS^B_{XY} = \frac{P_X}{P_Y}.$$

<u>Utility Possibility Frontier</u>

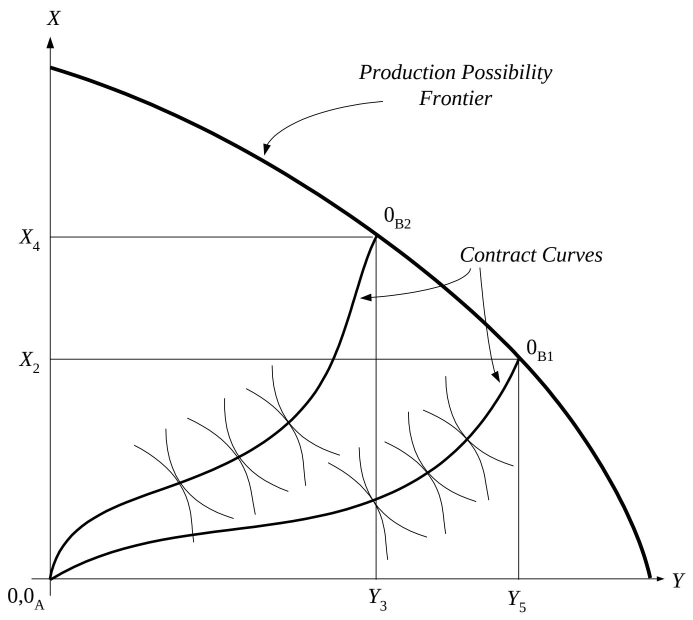
Note that the contract curve for the Edgeworth box in consumption space plots out all the combinations of utility levels for *A* and *B* where the utility of *A* is maximized given the utility of *B* and visa versa. Hence, as with the *ppf*, we can plot out the loci of points represented by the contract curve in utility space where the axes are the utilities for *A* and *B*, that is we can produce a utility possibilities frontier, *upf*.

Figure 13 Utility Possibility Frontier



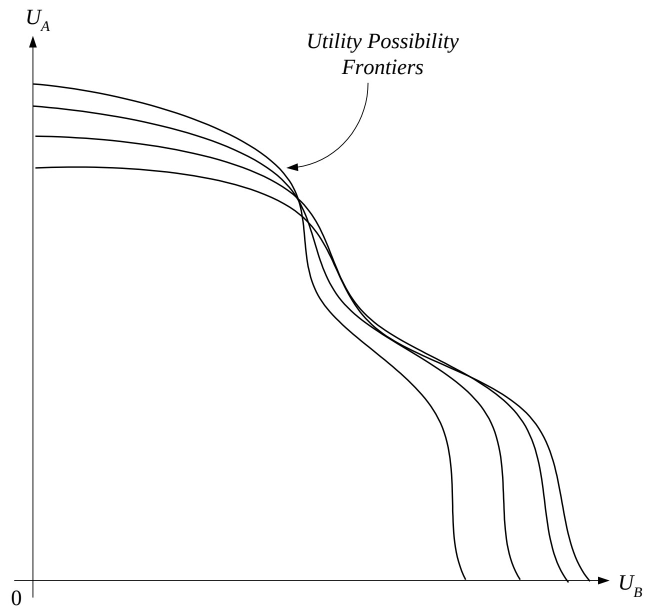
Notice however that each point on the *upf* is only defined for the distribution of a specific bundle of goods between the two consumers, since the dimensions of the Edgeworth box are fixed by reference to the predefined quantities of *X* and *Y* and therefore the contract curve is specific to those quantities of *X* and *Y*. We can see this simply by noting that the dimensions of the Edgeworth Box are determined by the quantities of *X* and *Y* that are available for distribution between the consumers *A* and *B*. Hence each of the possible Edgeworth Boxes in consumption corresponds to a point of the *ppf*, i.e.,

Figure 14 Production Possibility Frontier and Edgeworth Boxes in Consumption



Consequently, we can define a whole family of different *upf*s each relating to a specific combination of *X* and *Y*, i.e.,

Figure 15 Multiple Utility Possibility Frontiers

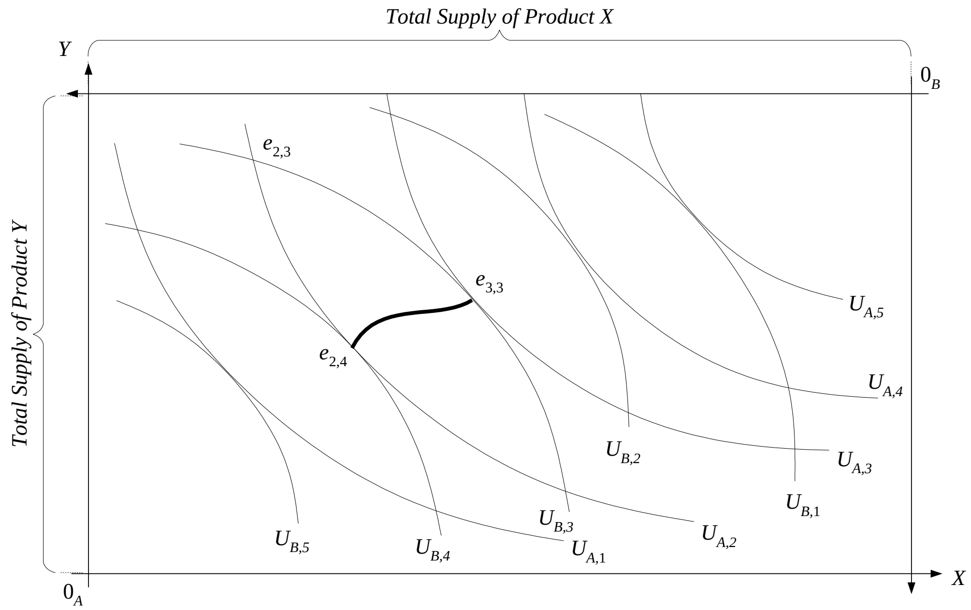


6. Simultaneous Equilibrium

So far we have developed the general equilibrium representations of production and consumption decisions largely independently. However, the fact that the *ppf* defines the range of possible dimensions for the Edgeworth Boxes in consumption suggests that there should be a link between the two representations. Furthermore, the very concept of general equilibrium requires that there should be some method by which the production and consumption sides of the economy are brought together. In fact, such a coincidence of decision-making has been ensured in the representations so far developed.

Consider the information contained in the prices paid by consumers. Consumers express their preferences through their willingness to pay for commodities, i.e., through the rates at which they are prepared to substitute one product for another - the *MRS*s. Beneficial exchange, i.e., exchange that makes both parties better off, can therefore continue so long as both of our consumers are prepared to exchange products *X* and *Y* because such exchanges make them better off. Consider again a situation in which our two consumers have initial endowments of two products that are off the contract curve.

Efficiency and Edgeworth Box in Consumption



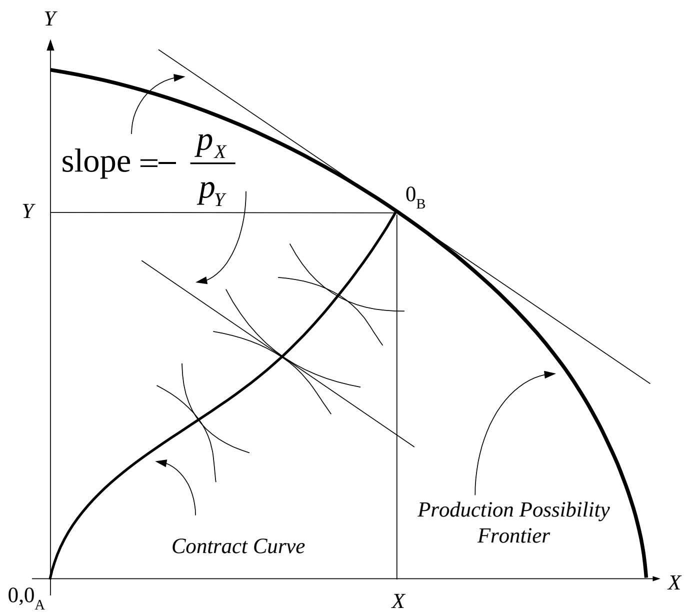
Starting from $e_{2,3}$, *A* and *B* can engage in mutually beneficial exchange. *A* can exchange *Y* for *X* and become better off and B can exchange *X* for *Y* and be better off, provided the exchanges leave them within the arcs described by the indifference curves X_2 and Y_3. The optimum will lie somewhere on the segment $e_{2,4}$ - $e_{3,3}$ of the contract curve. But note that, by definition, the contract curve is a locus of points where

$$\left(\frac{\partial U/\partial X}{\partial U/\partial Y}\right)^A = \left(\frac{\partial U/\partial X}{\partial U/\partial Y}\right)^B$$

$$MRS_{XY}^A = MRS_{XY}^B = \frac{p_X}{p_Y}$$

and hence it is the interaction of the preferences of the consumers that determines the relative prices of X and Y. But the relative prices of X and Y determine the allocatively efficient point on the *ppf*, i.e., the loci of technically efficient points. Hence we get the following illustration for simultaneous or general equilibrium in production and consumption.

General Equilibrium



This is intuitively sensible since consumers have no incentive to change their consumption patterns and producers have no incentive to change their production patterns. It is also consistent with the argument that production takes place to satisfy the demands of consumers and that the demand for factors of production is derived from the demand for products by consumers.

Conditions for Simultaneous Efficiency

Formally simultaneous, or general, equilibrium requires that

$$MRPT_{XY} = MRS_{XY}^{A} = MRS_{XY}^{B}.$$

We have already shown that the production optimum yields

$$MRPT_{XY} = \frac{p_X}{p_Y}$$

and that the consumption optimum yields

$$MRS_{XY}^{A} = MRS_{XY}^{B} = \frac{p_X}{p_Y}$$

and therefore

$$MRPT_{XY} = MRS_{XY}^{A} = MRS_{XY}^{B}.$$

General equilibrium requires the simultaneous realisation of three conditions

$$MRTS_{KL}^{x} = MRTS_{KL}^{y} = \frac{w}{r}$$

$$\frac{MC_x}{MC_y} = \frac{P_x}{P_y} = MRPT_{yx}$$

$$MRS_{xY}^{A} = MRS_{XY} = \frac{P_X}{P_Y}.$$

7. The Allocation of Resources in General Equilibrium

We have shown so far the process by which quantities are determined in general equilibrium, i.e., the quantities of factors used in the production of each good and the quantities of each good produced and then the quantities of each good consumed by each consumer. We need to take a brief look at the determination of prices and the implications for income distribution.

Prices for Products and Factors

We need to determine 4 prices; the wage rate w, the rental price of capital r and the prices of the products p_X and p_Y. But we only have three independent relationships.

i) Profit maximisation requires that

$$MRTS_{LK}^{x} = MRTS_{LK}^{y} = \frac{w}{r} = MRTS_{LK} \tag{1}$$

ii) Utility maximisation requires that

$$MRS_{XY}^{A} = MRS_{XY} = \frac{P_X}{P_Y} \tag{2}$$

iii) The optimal levels of factor use require that

$$w = MPP_L^X.p_X = MPP_L^Y.p_Y. \tag{3a}$$

$$r = MPP_K^X.p_X = MPP_K^Y.p_Y. \tag{3a}$$

But the equations 3a and 3b are not independent as can be shown by dividing them by each other, which produces

$$\frac{w}{r} = \frac{MPP_L^X.p_X}{MPP_K^X.p_X} = \frac{MPP_L^Y.p_Y}{MPP_K^Y.p_Y} = MRTS_{L,K}. \tag{4}$$

Hence we have a slight problem; 3 equations and 4 unknowns. Let one price, say p_X, be fixed, which is known as the *numéraire*. From (1)

$$w = r.\left(MRTS_{LK}\right) \tag{5}$$

and from (3b)

$$r = MPP_K^X.p_X \tag{6}$$

and combining (5) and (6) gives

$$w = \left(MPP_K^X.p_X\right).\left(MRTS_{LK}\right) \tag{7}$$

and rearranging (2) gives

$$p_Y = MRS_{XY}.p_X. \tag{8}$$

Equations (6), (7) and (8) define the three prices in the system, other than the *numéraire*, relative to the *numéraire* and in terms of known parameters, i.e.,

$$\frac{r}{p_X} = \left(MPP_K\right)$$

$$\frac{w}{p_X} = \left(MPP_K^X\right).\left(MRTS_{LK}\right)$$

$$\frac{p_Y}{p_X} = \left(MRS_{XY}\right)$$

and hence our system only solves for relative prices.

Income Distribution

So far we have avoided any mention of how much of *X* and *Y* go to *A* and *B*. The obvious determinants of these relationships are the levels of income received by *A* and *B*, and equally clearly these depend upon the prices paid for the factors and the quantities of the factors owned by each household. But we have already seen that the system only solves for relative prices, and therefore the problem reduces to the quantities of factors owned by each household.

The circular flow requires that total expenditure equals total income in the system, i.e.,

$$p_X.X + p_Y.Y = w.\bar{L} + r.\bar{K} \tag{9}$$

and for each consumer total expenditure must also equal total income, i.e.,

$$p_X.X_A + p_Y.Y_A = w.\bar{L}_A + r.\bar{K}_A \tag{10a}$$

$$p_X.X_B + p_Y.Y_B = w.\bar{L}_B + r.\bar{K}_B \tag{10b}$$

and the assumption of full employment ensures that

$$\bar{L} = L_A + L_B \quad \text{and} \quad \bar{K} = K_A + K_B \tag{11}$$

which seems to be 5 equations in four unknowns, where the unknowns are the quantities of labour and capital owned by each household.

But equations (10a) and (10b) are not independent, by the product exhaustion theorem (see Appendix 1). We can solve this by setting one quantity as a *numéraire* but, unlike the case with prices, this is not a value free presumption – why should an individual's endowment of any factor be fixed, whereas for prices we are dealing with means to ends.

Different distributions of resources will lead to different product combinations and therefore to different general equilibrium solutions. This is critical to the considerations of welfare economics.

8. Society's (Programming) Problem

In essence, what we have developed is a programming problem whereby society seeks to maximize its welfare subject to a series of constraints, i.e.,

$$\max_{X,Y} W = W\left(u^A, u^B\right)$$

subject to

$$\begin{array}{lll} X = X^A + X^B = x\left(L^x, K^x\right) \quad Y = Y^A + Y^B = y\left(L^Y, K^Y\right) & & \text{technology} \\ u^A = u^A\left(X^A, Y^A\right) \qquad u^B = u^B\left(X^B, Y^B\right) & & \text{preferences} \\ \bar{L} = L^x + L^y \qquad \bar{K} = K^x + K^y & & \text{factor endowments} \end{array}$$

This presentation has the advantage of being general but the disadvantage of being insufficiently specific to be convertible into a practical model.

One way forward is to give specific functional forms to technology

$$X = \alpha_x L_x^{\beta_1} K_x^{\beta_2} \quad Y = \alpha_y L_y^{\beta_3} K_y^{\beta_4}$$

and preferences

$$U_a = \alpha_a X_a^{\gamma_1} Y_a^{\gamma_2} \quad U_b = \alpha_b X_b^{\gamma_3} Y_b^{\gamma_4} .$$

Provided the coefficients sum to one each component of the system will be linear homogenous.

But specific functional forms alone are not enough for an empirical economic model. For that we also need data, which can be presented in a SAM format, e.g.,

Table 1 Social Accounting Matrix for a Simple General Equilibrium Model

		Commodities		Activities		Factors		Households		
		Primary	Secondary	Agriculture	Industry	Labour	Capital	Urban	Rural	Total
Commodities	Primary							50	75	125
	Secondary							100	50	150
Activities	Agriculture	125								125
	Industry		150							150
Factors	Labour			62	55					117
	Capital			63	95					158
Households	Urban					60	90			150
	Rural					57	68			125
Total		125	150	125	150	117	158	150	125	

This allows us to define a general equilibrium problem as an empirical problem of optimization. This is the problem that we will learn how to programme in the rest of this part of the course. Initially the problem will be limited to the 2*2*2*2 model with no government, savings/investment and trade. We will then relax the assumptions of no government and savings/investment to introduce policy instruments, and then in the next part of the course we will introduce trade.

Appendices

Product Exhaustion Theorem

This relates to the controversies about marginal productivity theory and whether the theory satisfies the accounting identity requirement that

$$\begin{gathered}\begin{pmatrix}\text{value of}\\ \text{product}\end{pmatrix}=\begin{pmatrix}\text{cost of}\\ \text{labour}\end{pmatrix}+\begin{pmatrix}\text{cost of}\\ \text{capital}\end{pmatrix}.\\ \left(p*X\right)=\left(w*L\right)+\left(r*K\right)\end{gathered}\tag{1}$$

To satisfy this identity it is necessary that the determinants of *w* and *r* are such that the sum of the shares of the product going to labour and capital equal one. Dividing through (1) by (*p***X*) gives

$$\begin{gathered}1=\frac{\left(w*L\right)}{\left(p*X\right)}+\frac{\left(r*K\right)}{\left(p*X\right)}\\ 1=\begin{pmatrix}\text{share of}\\ \text{labour}\end{pmatrix}+\begin{pmatrix}\text{share of}\\ \text{capital}\end{pmatrix}\end{gathered}.\tag{2}$$

The marginal productivity theory of the determination of factor pricing specifies that factors should be paid the value of their marginal product, i.e.,

$$p*X=\left(MPP_L.p\right)*L+\left(MPP_K.p\right)*K\tag{3}$$

and hence that

$$X=\left(MPP_L\right)*L+\left(MPP_K\right)*K\tag{4}$$

which means that if factors are paid the value of their marginal products then payments to factors will exhaust the product.

<u>Proof using Euler's Theorem</u>

In the general case, assume a production function that is homogenous of degree *v*, i.e.,

$$f\left(\lambda L,\lambda K\right)=\lambda^{v}.f\left(L,K\right)\tag{5}$$

and differentiate with respect to λ

$$L.\frac{df}{dL}+K.\frac{df}{dk}=\nu.\lambda^{\nu-1}.f\left(L,K\right) \tag{6}$$

and if v =1, (6) simplifies to

$$L.\left(MPP_L\right)+K.\left(MPP_K\right)=f\left(L,K\right)=X\ . \tag{7}$$

Thus, the total real payment to factors will exhaust the physical product if there are constant returns to scale. The monetary equivalent is achieved by multiplying through (7) by p, i.e.,

$$p*X=L.\left(MPP_L*p\right)+K.\left(MPP_K*p\right) \tag{8}$$

such that factors are paid the value of their marginal products.

The special case of a Cobb-Douglas production function is similarly straightforward. The production function is

$$X=a.L^{\alpha}K^{\beta} \tag{9}$$

the *MPP*s for labour and capital are

$$\frac{\partial X}{\partial L}=\alpha a.L^{\alpha}K^{\beta}L^{-1}=\alpha.\frac{X}{L}\qquad\text{and}\qquad\frac{\partial X}{\partial K}=\chi a.L^{\alpha}K^{\beta}K^{-1}=\beta.\frac{X}{K}\ . \tag{10}$$

With perfect factor and product markets the factor payment rates are

$$w=\alpha.\frac{X}{L}.p\qquad\text{and}\qquad r=\beta.\frac{X}{K}.p \tag{11}$$

and therefore, the product will be exhausted if

$$p*X=\left(w*L\right)+\left(r*K\right). \tag{12}$$

Substituting (11) into (12) gives

$$\begin{aligned}p*X&=\left(\alpha.\frac{X}{L}.p*L\right)+\left(\beta.\frac{X}{K}.p*K\right)\\ p&=\left(\alpha*p\right)+\left(\beta*p\right)\\ 1&=\alpha+\beta\end{aligned} \tag{13}$$

and demonstrates that provided the Cobb-Douglas production function has constant returns to scale then the product exhaustion theorem holds.

Euler's theorem is an identity and therefore the product exhaustion theorem holds for all values of the variables in linear homogenous, i.e., constant returns to scale, production and utility functions.