A Single Region Computable General Equilibrium Model: Online Course Version (*smod_t/2/3*)

cgemod

Abstract

This document reports on three variants of the smod suite of simple CGE models that are calibrated using data derived from the GTAP database for the purposes of training and model development. The data have been 'adjusted' to fit with the pedagogic purposes of the smod suite: hence the data are not suitable for 'real-world' policy analysis.[1] *These variants have been devised specifically for online course training and are supplied with sample data sets for several regions with different commodity and activity aggregations. Trade is modelled using the Armington insight, with CES and CET functions, for a single country trading with an aggregated rest of the world; imports are valued cif and exports are valued fob. Production is modelled using Cobb-Douglas or nested CES functions. Household utility functions are Linear Expenditure Systems with two households; the data for the household accounts have been mathematically generated. The government has five tax instruments. A companion document provides a series of structured exercises. All the computer code and data are stored as a GAMS User Library.*

1 In fact, as noted by GTAP, it is NOT appropriate to use GTAP data for single countries to conduct real-world policy analyses.

Contents

1. Introduction

The *smod_t* models are simply single country CGE models that are calibrated with data drawn from the Global Trade Analysis Project (GTAP) database. The model is a derivative of the class of single country CGE models described by Dervis *et al*., (1984). It is designed as a platform for learning about CGE models and **not** as a tool for the conduct of (policy) experiments designed to influence policy choices. The depth of institutional information contained in the SAM is limited, e.g., there are no accounts for incorporated business enterprises, and the data have been 'adjusted' to fit the pedagogic purposes of this course. The model is supplied with data for several regions with different commodity and activity aggregations. The data have been extracted from a reduced form version of the GTAP database and then mathematical adjusted to meet the learning objectives of the course.[2] Full functionality of the model is available within the bounds of demo GAMS.[3]

The basic *smod_t* model combines features from a simple closed economy model with the behavioural equations for trade derived from the 123 model. The functional forms used are Cobb Douglas (CD) for production and CES/CET for trade; the utility function used is Stone-Geary (linear expenditure system), which is a development of the CD utility function used in the closed economy model.

The *smod_t2 and smod_t3* models introduce the use of nested CES production function: *smod_t2* uses a two-level nest using CES functions with an option of Leontief at the top level, while *smod_t3* extends the production system to a three-level CES nest for primary inputs. The three-level CES used in *smod_t3* can be generalised to an n-level CES nest that encompasses primary inputs and intermediate inputs with minor additions to the code, i.e., provides the basis for variants used to model energy, water use, climate change, etc. The other behavioural; features of *smod_t* are carried over into *smod_t2* and *smod_t3*.

The notes for these models proceed by first describing the data used by the model. This is laid out as a SAM. The SAM structure is also used to identify the behavioural and transaction relationships of the model. These are followed by schematic representations of the price and quantity systems of the *smod_t* model, and then a formal algebraic statement of the

2 The data are from version 8 of the GTAP database, which has 57 commodity and activity accounts, 5 factor accounts and divides the world into 129 regions. The data supplied with the model are aggregates that include multiple regions and have been 'adjusted'. See Appendix 1 for further details of the data.

3 A demo license for the GAMS software is often available from sales@gams.com

model's equations. The equations are then summarised in a table that also identifies the equation names used in the GAMS code and the equation and variable counts for the model.

The *smod_t2/3* variants introduce the use of nested CES functions in CGE models. In these variants the focus is on production and only equations, and associated variables and parameters, need to be changed. Accordingly, the model changes only need to be detailed for the production block. The *smod_t2* variant is a two level CES nest, which is detailed in Appendix 2, while the *smod_t3* variant is a three level CES nest detailed in Appendix 3.

2. Social Accounting Matrix

The GTAP database is primarily used to analyse changes in trade policies on a global scale; hence the distinctive feature of the database is the richness of information about bi lateral trade transactions (see McDonald and Thierfelder (2019)[4], which is a revised version of GTAP Technical Paper 22, for a detailed description of the GTAP database in a SAM format).[5] This reduced form of the database has been stripped of all bi lateral trade data, such that for each region there is a single trade account with exports valued free on board (*fob*) and imports valued carriage, insurance and freight (*cif*) paid, and with a simplified treatment of taxes on commodity purchases. However, it has been 'augmented' with data on the government budget surplus/deficit, which allows for estimates of savings and direct taxes by the households (see McDonald and Sonmez, 2004); this makes for a somewhat more realistic treatment of the government account.

Table 1 A Macro Social Accounting Matrix for the CGE Model ($(USD) bn)

	Commodities	**Activities**	**Factors**	**Households**	**Factor Taxes**	**Govt**	**Investment**	**Rest of World**
Commodities		3,849.1		2,588.1	750.4	880.6	1,000.2	
Activities	7,758.1							7,758.1
Factors		3,278.9						
Households			2,827.3					
Factor Taxes	240.8	630.0		511.1				
Government			451.7	-271.9	631.5		69.3	
Savings	1,069.5							1,069.5
Rest of World	9,068.4	7,758.1	3,278.9	2,827.3	1,381.9	880.6	1,069.5	9,068.4
Totals		3,849.1		2,588.1	750.4	880.6	1,000.2	

Source: Augmented GTAP database version 8 for OECD

The macro SAM for OECD reported in Table 1 indicates the coverage of transactions provided by these SAMs: the model will encompass transactions for all the non-zero cells, whereas non-zero transactions in other cells will cause the model to fail (Note: the row totals are omitted to save space). There are five types of commodity demand – intermediate inputs, final demand by households, government, investment, and export demand – that are supplied

4 www.cgemod.org.uk/Deriving a Global GTAP SAM cgemodTP 12.pdf

5 This is now part of the GTAPAgg suite that comes with a GTAP licence.

from two sources – domestic productions and imports. However, there are also a variety of tax instruments on the commodity accounts; these are import duties, export taxes and sales taxes.

Domestic production activities use intermediate and primary inputs and pay production taxes and factor use taxes to the government. Factor incomes are earned by the sale of factor services to domestic activities and the incomes from factor services accrue to domestic households. Household incomes are used to pay direct (income) taxes, to save and to pay for commodities.

Thus, the government receives income from five types of tax – import duties, export taxes, sales taxes, production taxes, and income taxes – and spends this income on commodities and net savings. The rest of the world provides a third source of savings in the form of the balance on the investment-savings account; all savings are spent to purchase commodities for investment.

Behavioural Relationships

While the SAM provides details of current account transactions, and the agents involved in those transactions it only provides limited information about the behavioural relationships. The behavioural relationships adopted in this variant of the model are simple. The activities are assumed to maximise profits using Cobb-Douglas technology to produce aggregate primary inputs and Leontief technology to produce aggregate intermediate inputs. The household maximises utility subject to preferences represented by a Stone-Geary utility functions (Linear Expenditure System), having first paid income taxes and saving a proportion of after-tax income.

The Armington insight is used to model trade. Domestic output is distributed between the domestic market and exports according to a constant elasticity of transformation (CET) function, while domestic absorption is satisfied from domestic production and imports that are mixed to provide a composite commodity according to a CES function. The optimal ratios of imports to domestic commodities and exports to domestic commodities are determined by first order conditions based on relative prices. The small country trade assumption applies to all imports and exports.

Table 2 Behavioural Relationships for the CGE Model

	Commodities	Activities	Factors	Households	Government	Investment	Rest of World	Totals	
Commodities	0	Leontief Input-Output Coefficients	0	Linear Expenditure System	Fixed Exogenously	Fixed Shares of Savings	Commodity Exports	Commodity Demand	Commodity Price, Export prices
Activities	C-D Production Functions	0	0	0	0	0	0	Domestic Production	Activity Prices
Factors	o	Factor Demands	0	0	0	0	0	Factor Income	
Households	0	0	Fixed Shares of Factor Income	0	0	0	Net Remittances	Household Income	
Government	Revenue from Sales Taxes, Import Duties and Export Taxes	Production Taxes	0	Direct Taxes on Household Income	0	0	Net Grants/Aid	Government Income	
Savings	0	0	Depreciation	Household Savings	Government Savings (Residual)	0	Current Account 'Deficit	Total Savings	
Rest of World	Commodity Imports	0	0	0	0	0	0	Total 'Expenditure' Abroad	
Totals	Commodity Supply	Activity Inputs	Factor Expenditure	Household Expenditure	Government Expenditure	Total Investment	Total 'Income' from Abroad		
	Producer Commodity Prices Domestic and World Prices for Imports	Value Added Prices							

Table 3 Transactions Relationships for the CGE Model

	Commodities	Activities	Factors	Households	Government	Investment	RoW
Commodities	0	$\left(PQD_c * ioqint_{c,a} * QINT_a\right)$	0	$\left(\sum_h PQD_c * QCD_{c,h}\right)$	$\left(PQD_c * QGD_c\right)$	$\left(PQD_c * QINVD_c\right)$	$\left(PWE_a * QE_a * ER\right)$
Activities	$\left(PDS_c * QDS_c\right)$	0	0	0	0	0	0
Factors	0	$\left(WF_f * WFDIST_{f,a} * FD_{f,a}\right)$	0	0	0	0	
Households	0	0	$\sum_f \left(hovash_{h,f} * YF_f\right)$	0	0	0	$howor_h * ER$
Government	$\left(tm_c * PWM_c * QM_c * ER\right)$ $\left(te_a * PWE_a * QE_a * ER\right)$ $\left(ts_c * PQ_c * QQ_c\right)$	$\left(tx_a * PX_a * QX_a\right)$	0	$\left(ty * YH\right)$	0	0	$govwor * ER$
Savings	0	$deprec_f * YF_f$	0	$\left(shh_h * YH\right)$	$\left(YG - EG\right)$	0	$\left(KAPWOR * ER\right)$
Rest of World	$\left(PWM_c * QM_c * ER\right)$	0	0	0	0	0	0
Total	$\left(PQ_c * QQ_c\right)$	$\left(PX_a * QX_a\right)$	YF_f	YH_h	YG	$INVEST$	0

All commodity and activity taxes are expressed as simple *ad valorem* tax rates, while income taxes are defined as a fixed proportion of household income[6], i.e., in all cases the average and marginal rates are assumed equal. Import duties and export taxes apply to imports and exports, while sales taxes are applied to all domestic absorption, i.e., imports are subject to sequential import duties and sales taxes. Production taxes are levied on the value of output by activity. Income taxes are taken out of household income and then the households are assumed to save a proportion of disposable income. This proportion is either fixed or variable according to the closure rule chosen for the investment-savings account.

Government expenditure consists of commodity (final) demand, which is assumed to be fixed in real terms, and expenditure on any subsidies, i.e., negative taxes. Hence government saving, or the internal balance, is defined as a residual. However, the closure rules for the government account allow for various permutations. In the base case it is assumed that the tax rates and volume of government demand is variable, with government savings calculated as a residual. However, the tax rates can all be scaled equiproportionately using tax specific scaling factors; hence for instance the value of government savings can be fixed and one of the tax scalars can be made variable thereby producing an estimate of the constrained optimal tax rate. If the analyst wishes to change the relative tax rates across commodities (for import duties, export taxes and sales taxes) or across activities (for production taxes) then the respective tax rate parameters can be altered. Equally the volume of government consumption can be changed by adjusting the closure rule with respect to the scaling adjuster attached to the volumes of government consumption. The pattern of government expenditure is altered by changing the parameter that controls the pattern of government expenditure (*qgdconst*).

Total savings come from the factors (depreciation), households, enterprises, the internal balance on the government account and the external balance on the trade account. The external balance is defined as the difference between the value of total exports and total imports, converted into domestic currency units using the exchange rate.[7] In the base model it is assumed that the external balance is fixed and that the exchange rate is variable. Alternatively, the exchange rate can be fixed, and the external balance can be variable.

6 This is in effect an *ad valorem* tax rate.

7 Note, in the code, we calculate the external balance as the value of imports minus the value of exports, converted into domestic currency units using the exchange rate. Then, when a country is running a trade deficit, the calculation of the external balance is positive and that term is added to the total savings, it is the borrowing from abroad.

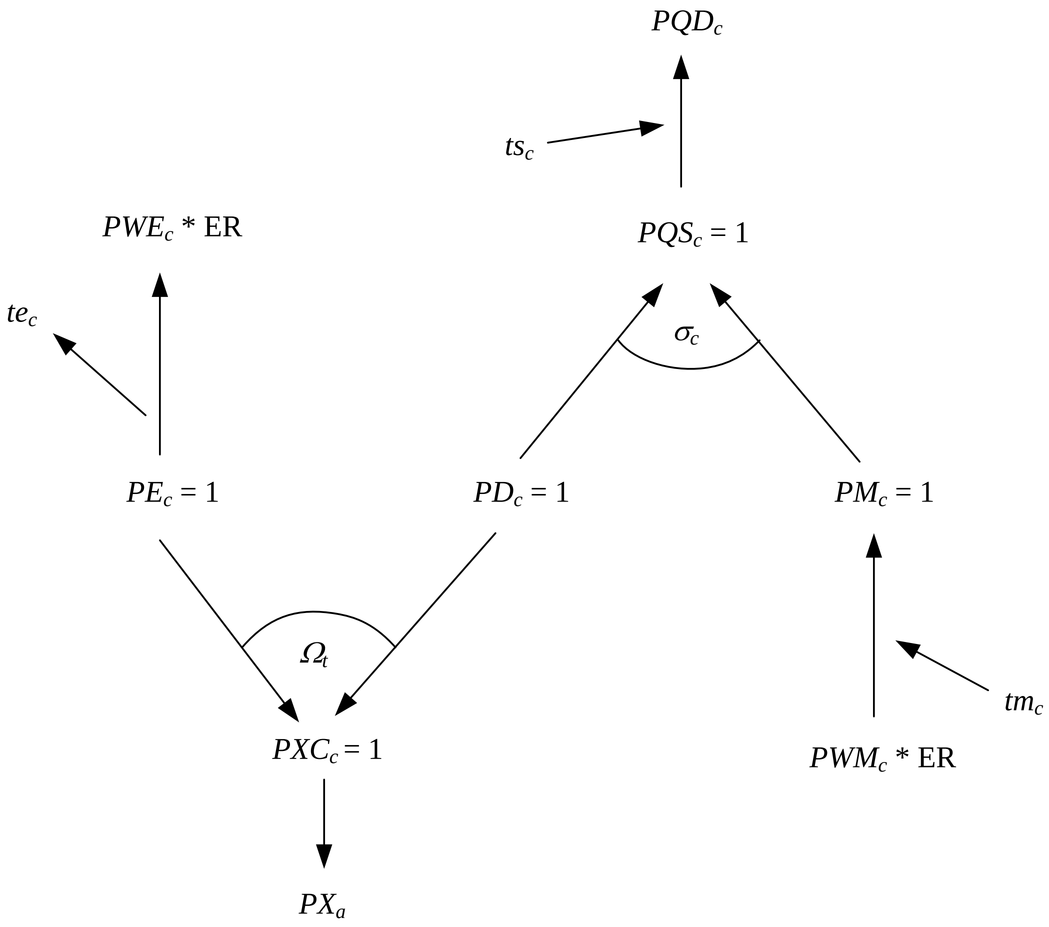
Expenditure by the investment-savings account consists solely of commodity demand for investment. In the base solution it is assumed that the volume of investment adjusts so that total expenditure on investment is equal to total savings, i.e., the closure rule presumes that savings drive investment expenditures. The pattern of investment volumes is fixed, and hence the volume of each commodity changes equiproportionately according to the volume of savings. It is possible to fix the volume of real investment and then allow the savings rate by households to vary to maintain balance in the investment-savings account, and it is possible to change the patterns of investment by changing the investment parameters (*qinvdconst*).

Commodity Price and Quantity System

The price system is built up using the principle that the components of the 'price definitions' are the entries in the columns of the SAM. Hence there are a series of explicit accounting identities that define the relationships between the prices and thereby determine the processes used to calibrate the tax rates for the base solution. However, the model is set up using a series of linear homogeneous relationships and hence is only defined in terms of relative prices. Consequently, as part of the calibration process, it is necessary to set some of the prices equal to one (or any other number that suits the modeller) – this model, and all cgemod models, adopts the convention that prices are normalised at the level of the CES and CET aggregator functions (see Figure 1), which means they are normalised on basic prices as defined in the System of National Accounts (SNA).

The relationships between the various commodity prices in the model are illustrated in Figure 1. The domestic producer/basic prices (*PQS*) are CES aggregates of the domestic prices of the domestically supplied commodities (*PD*) and the domestic prices of the imports (*PM*) under the maintained assumption that the domestic and imported variants of the commodities are imperfect substitutes, with a constant elasticity of substitution (σ_c). The differences between producer/basic and consumer/purchaser prices (*PQD*) are defined by the *ad valorem* sales tax rates (*ts*) that are levied on all domestic demand. The import prices are defined by the world prices in foreign currency units (*PWM*), the exchange rate (*ER*) and the *ad valorem* import tariff rates (*tm*). The commodity prices (*PXC*) received by domestic producers are CET aggregates of the domestic prices (*PD*) and the export prices (*PE*), with a constant elasticity of transformation (Ω_t). The export prices being defined by the world prices in foreign currency units (*PWE*), the exchange rate (*ER*) and the *ad valorem* export subsidy rates (*te*). This is a standard application of the Armington insight on the import side; while the application on the export side, is standard, it is more contentious.

Figure 1 Commodity Price Relationships for the CGE Model

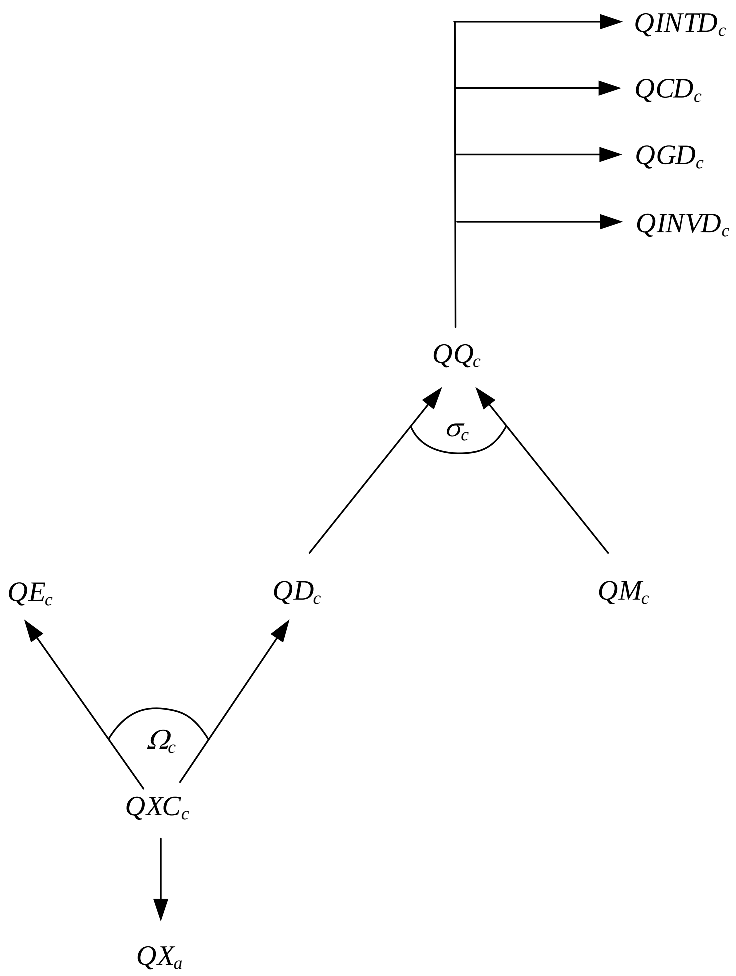


Finally, the average prices received by activities (*PX*) are the same as the commodity prices (*PXC*) received by domestic producers since each activity only produces a single commodity and each commodity is only produced by a single activity. This ensures that the commodity price system passes a series of prices down to the production side of the economy, which is explored in more detail below.

Figure 2 illustrates the quantity relationships for commodities. Domestic demand (*QQ*) is made up of intermediate demand (*QINTD*), household consumption (*QCD*), government consumption (*QGD*) and investment (*QINVD*). While domestic supply is satisfied by domestic production sold on the domestic market (*QD*) and imports (*QM*). Domestic commodity production (*QXC*) is divided between the domestic market (*QD*) and exports (*QE*). Since each activity only produces a single commodity and each commodity is only produced by a single activity, domestic commodity production is defined by the outputs of each activity (*QX*).

Note how the application of the 'Law Of One Price' (LOOP) requires that all domestic agents purchase commodities at the same price (*PQD*); hence *QINTD*, *QCD*, *QGD* and *QINVD* are all transactions that take place at a common price.

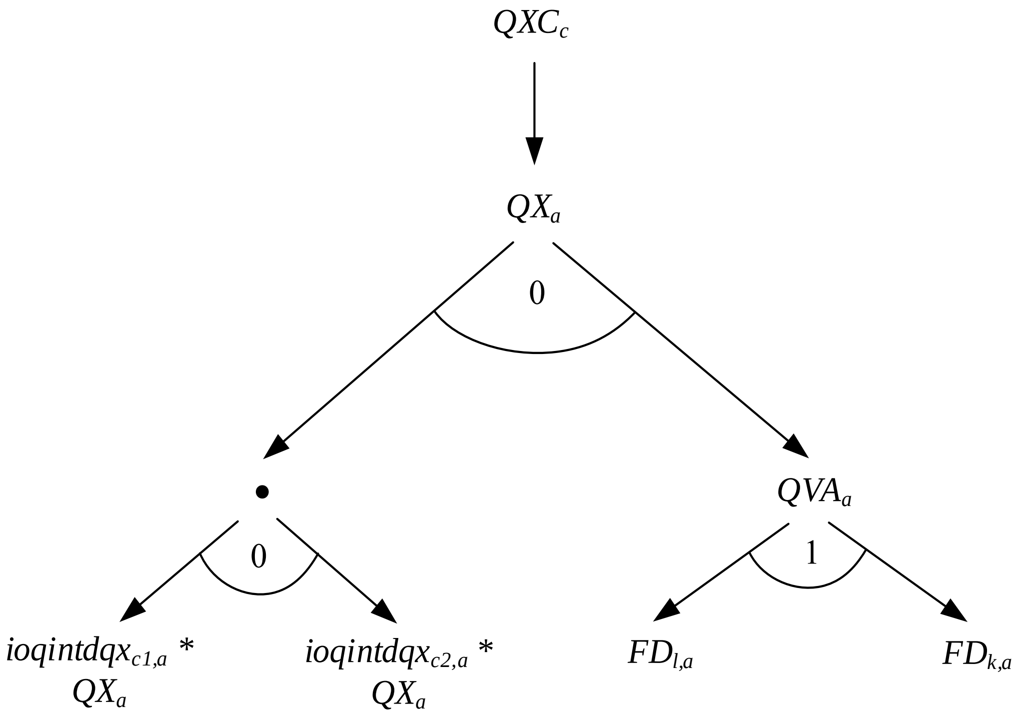
Figure 2 Quantity Relationships for a Basic CGE Model



<u>Production Price and Quantity System</u>

The domestic production relationships are illustrated in Figures 3 and 4 under the simplifying assumptions of two intermediate inputs and two primary inputs. Production is a Cobb-Douglas aggregate of factor inputs (*FD*). Intermediates in production use Leontief input-output coefficients (see Figure 3).

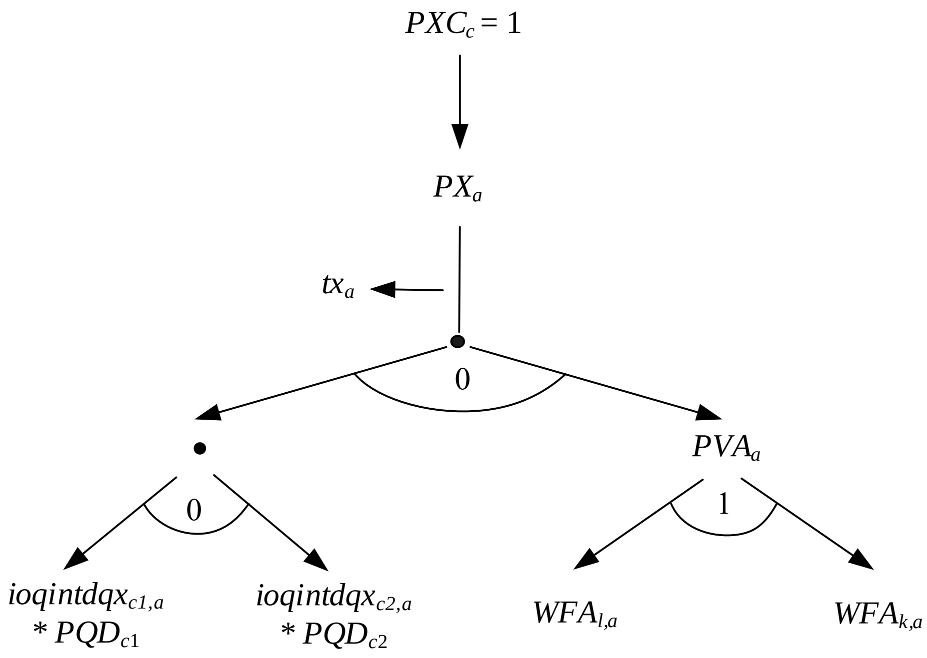
Figure 3 Production Quantity Relationships for the CGE Model



There is a matching price system (Figure 4). The unit price received by an activity (*PX*) is defined by the composite price for the matching commodity (*PXC*). This unit price must then be allocated between payments for intermediate inputs, primary inputs, and taxes on production. First, the activity price (*PX*) is reduced by the amount of the *ad valorem* production tax rate (*tx*) and then the residual is divided between payments to primary and intermediate inputs. Intermediate inputs are treated very simply; it is assumed that a unit of activity output uses a fixed proportionate quantity of each intermediate input, i.e., Leontief input-output technologies are assumed, and hence expenditure on intermediate inputs per unit of output is the sum of the intermediate input prices multiplied by the technical coefficients derived from the Use matrix. This is still a common presumption in the literature, and represents a convenient and simple starting point, especially since it allows a simple redefinition of the price of value added (*PVA*). This means that there is a 'wedge' between activity prices (*PX*) and value-added prices (*PVA*) that must be included in the model. The optimal quantities of each factor are then determined by first order conditions based on average factor prices (*WF*).[8]

[8] The model allows for factors that are notionally the same to be paid different rates according to the activity that employs them by using a fixed set of proportionate differences from the average factor prices (*WFDIST*); *de facto* these adjustments operate as activity specific differences in the productivity that are

Figure 4 Production Price Relationships for the CGE Model



solely attributable to the activity. In the base model, these adjustments are fixed at one. In the base data, there is no information on physical quantities of inputs used in production. Therefore, the inputs are set equal to the value payments to inputs and the average factor prices (*WFDIST)* are equal to one. If there are data on factor use by activity, the average factor prices will differ from one.

3. Algebraic Statement of a Single Region CGE Model

Model Sets

Rather than writing out every equation in detail it is useful to start by defining a series of sets. Then if a behavioural relationship applies to all members of a set an equation only needs to be specified once. The natural choice of sets for this model is commodities, activities, and factors, which are define as

$$c = \{\text{cagr, cnres, cmanu cserv, }\}$$
$$a = \{\text{aagr, anres, amanu, aserv, }\}$$
$$f = \{\text{flnd,fuskl,Ffsklb,fcap}\}$$
$$h = \{\text{h\_rich, h\_poor}\}$$
$$g = \{\text{imptax,exptax,saltax,prodtax,dirtax,govt}\}$$
$$i = \{i\_s\}$$
$$w = \{\text{row}\}$$

Various subsets of *c* are declared and then assigned based on certain characteristics of the data set used to calibrate the specific implementation of the model, so-called dynamic sets. These subsets are

$$ce(c) = \text{export commodities}$$
$$cen(c) = \text{non-export commodities}$$
$$cm(c) = \text{imported commodities}$$
$$cmn(c) = \text{non-imported commodities}$$
$$cx(c) = \text{commodities produced domestically}$$
$$cxn(c) = \text{commodities NOT produced domestically AND imported}$$
$$cd(c) = \text{commodities produced and demanded domestically}$$
$$cdn(c) = \text{commodities NOT produced and demanded domestically}$$

$$acx(a) = \text{activities purchased domestically}$$
$$acxn(a) = \text{activities NOT purchased domestically}$$

Note how for each set, e.g., *cm*, a complement is declared and assigned, e.g., *cmn*. Throughout any set label ending in '*n*' is the complement to the set identified by the characters before the '*n*'.

It is also necessary to define a global set, *sac*, as

$$sac = \{c, a, f, h, g, i, w, total\}.$$

A macro SAM that can be used to check model calibration is very useful. This needs another set

$$ss = \{commdty, activity, valuad, hholds, govtn, kapital, world, totals\}.$$

Conventions

The equations for the model are set out in 'blocks'. The order in which you proceed is largely a matter of personal preference; for this model the blocks used are for 'exports', 'imports', 'commodity prices', 'numéraire prices', 'production', 'factor', 'household', 'taxes', 'government', 'kapital (savings and investment)', 'market clearing' and 'model closure'. A series of conventions are adopted for the model.

- All VARIABLES are in upper case.
- The standard prefixes for variable names are: *P* for price variables, *Q* for quantity variables, *W* for factor prices, *F* for factor quantities, *E* for expenditure variables, *Y* for income variables, and *V* for value variables
- All variables have a matching parameter that identifies the value of the variable in the base period. These parameters are in upper case and carry a '0' suffix, and are used to initialise variables.
- A series of variables are declared that allow for the equiproportionate multiplicative adjustment of groups of variables. These variables are named using the convention ***ADJ*, where ** indicates the variable series they adjust.
- All parameters are in lower case, except those paired to variables that are used to initialise variables.
- Parameter names have a two or five character suffix which distinguishes their definition, e.g., ***sh* is a share parameter, ***av* is an average and ***const* is a constant parameter.
- For the Armington (CES) functions all the share parameters are declared with the form *delta***, all the shift/efficiency parameters are declared with the form *ac***, and all the elasticity parameters are declared with the form *rho***, where ** identifies the function in which the parameter operates.

- For the CET functions all the share parameters are declared with the form *gamma***, all the shift/efficiency parameters are declared with the form *at***, and all the elasticity parameters are declared with the form *rho***, where ** identifies the function in which the parameter operates.
- All coefficients in the model are declared with the form *io*****, where **** consists of two parts that identify the two variables related by the coefficient.
- The index ordering follows the specification in the SAM: row, column.
- All sets have another name, or alias, given by the set name followed by "*p*". For example, the set of commodities may be called *c* or *cp*.
- All parameter and variable names have less than 10 characters.

Exports Block Equations

The domestic prices of exports (*PE*) are defined (X1) as the product of world prices of exports (*PWE*), the exchange rate (*ER*) and one minus the export subsidy rate[9] (*te*) multiplied by an export subsidy rate adjustment variable (*TEADJ*). The possibility of non-traded commodities means that the equations for the domestic prices of exports (and imports) are only implemented for those commodities that are traded; this requires the use of a dynamic set, *ce*, which is defined by those commodities that are exported in the base data. Also notice how the world prices of exports (*PWE*) are defined as variables; if the export prices are fixed, the classic 'small' country trade assumption is made.

Export Block Equations

$$PE_c = PWE_c * ER * \left(1 - \left(TEADJ * te_c\right)\right) \qquad \forall c \in ce \tag{X1}$$

$$QXC_c = at_c \left(\gamma_c * QE_c^{rhot_c} + (1-\gamma_c) * QD_c^{rhot_c}\right)^{\frac{1}{rhot_c}} \qquad \forall c \in \left(cd \cap ce\right) \tag{X2}$$

$$\frac{QE_c}{QD_c} = \left[\frac{PE_c}{PD_c} * \frac{(1-\gamma_c)}{\gamma_c}\right]^{\frac{1}{(rhot_c - 1)}} \qquad \forall c \in \left(cd \cap ce\right). \tag{X3}$$

$$QXC_c = QD_c + QE_c \qquad \forall c \in \left(cd \cap cen\right) \text{ or } \left(cdn \cap ce\right). \tag{X4}$$

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[9] Defining export taxes as negative subsidies means that there is symmetry between the treatment of import duties and export subsidies when coding the model in GAMS.

Domestic commodity outputs (*QXC*) are either exported (*QE*) or supplied to the domestic market (*QD*). The allocation of output between the domestic and export markets is determined by the output transformation functions, Constant Elasticity of Transformation (CET) functions, (X2), with the optimum combinations of *QE* and *QD* determined by first-order conditions (X3).[10]

However, some commodities may not be traded and therefore (X2) and (X3) are implemented only if the commodity is traded. This means that domestic commodity outputs are undefined for non-traded commodities, and the quantity produced (*QXC*) is either solely supplied to the domestic market (*QD*) or solely exported (*QE*): the conditions mean that a commodity is either produced and consumed domestically and not exported or produced domestically and exported but not consumed domestically (X4).

Imports Block Equations

The domestic prices of imports (*PM*) are defined (M1) as the product of world prices of imports (*PWM*), the exchange rate (*ER*) and one plus the import tariff rate (*tm*) multiplied by a tariff rate adjustment variable (*TMADJ*). The possibility of non-traded commodities means that the equations for the domestic prices of imports are only implemented for those commodities that are traded; this requires the use of a dynamic set, *cm*, which is defined by those commodities that are imported in the base data. Also notice how the world prices of imports (*PWM*) are defined as variables; if the import prices are fixed the classic 'small' country trade assumption is made.

However, both domestic and foreign producers can supply commodities to the domestic market. The composite (consumption) commodities are a mixture of imports (*QM*) and domestic demand (*QD*). The mixtures between the domestic and import supplies are determined by the substitution functions, Constant Elasticity of Substitution (CES) functions, (M2), with the optimal combinations of *QM* and *QD* being determined by first-order conditions (M3).[11]

[10] Note that second-order conditions for the CET functions are redundant given the properties of the functional form.

[11] Note that second-order conditions for the CES functions are redundant given the properties of the functional form.

However, some commodities are non-traded and therefore (M2) and (M3) are implemented only if the commodity is traded, which leaves *QQ* undefined for non-traded commodities. This means that commodities consumed domestically are either solely supplied from domestic production (*QD*) or solely imported (*QM*): the conditions mean that a commodity is either produced and consumed domestically and not imported or not produced domestically and imported (M4).

Import Block Equations

$$PM_c = PWM_c * ER * \left(1 + \left(TMADJ * tm_c\right)\right) \quad \forall c \in cm \tag{M1}$$

$$QQ_c = ac_c\left(\delta_c * QM_c^{-rhoc_c} + \left(1-\delta_c\right) * QD_c^{-rhoc_c}\right)^{-\frac{1}{rhoc_c}} \quad \forall c \in \left(cx \cap cm\right) \tag{M2}$$

$$\frac{QM_c}{QD_c} = \left[\frac{PD_c}{PM_c} * \frac{\delta_c}{\left(1-\delta_c\right)}\right]^{\frac{1}{(1+rhoc_c)}} \quad \forall c \in \left(cx \cap cm\right). \tag{M3}$$

$$QQ_c = QD_c + QM_c \quad \forall c \in \left(cx \cap cmn\right) \text{ or } \left(cxn \cap cm\right). \tag{M4}$$

<u>Commodity Price Block Equations</u>

The composite price equations (CP1, CP2 and CP3) are derived from the first order conditions for tangencies to consumption and production possibility frontiers. By exploiting Euler’s theorem for linearly homogeneous functions the composite prices can be expressed as expenditure identities rather than dual price equations for export transformation and import aggregation, such that *PQS* is the weighted average of the producer price of a commodity, when *PD* is the producer price of domestically produced commodities and *PM* the domestic price of imported commodity, (CP1), where *QD* the quantity of the domestic commodity demanded by domestic consumers, *QM* the quantity of imports and *QQ* the quantity of the composite commodity. Notice how the commodity quantities are the weights. This composite commodity price (CP1) does not include sales taxes, which create price wedges between the purchaser price of a commodity (*PQD*) and the producer (basic) prices (*PQS*). Hence the purchaser price is defined as the producer price plus the sales taxes (CP2). This formulation means that the sales tax is levied on all sales on the domestic market, irrespective of the origin of the commodity concerned.[12]

The composite output price for a commodity is also derived by exploiting Euler's theorem for linearly homogeneous functions, and is given by (CP3) where *PD* is the domestic producer price for the output of commodities supplied to the domestic market, *QD* is the supply of output to the domestic market, *QE* is the quantity exported by activities, *PXC* is the composite output price by commodity and *QXC* is the quantity of domestic production by commodity.

Commodity Price Block Equations

$$PQS_c = \frac{PD_c * QD_c + PM_c * QM_c}{QQ_c} \qquad \forall c \in (cd \cup cm) \tag{CP1}$$

$$PQD_c = PQS_c * \left(1 + \left(TSADJ * ts_c\right)\right) \qquad \forall c \in (cd \cup cm) \tag{CP2}$$

$$PXC_c = \frac{PD_c * QD_c + PE_{c \in ce} * QE_{c \in ce}}{QXC_c} \qquad \forall c \in cx \tag{CP3}$$

<u>Numéraire Price Block</u>

It is necessary to define a price numéraire; for this model two alternative indices that can serve as numéraire are defined so as to allow the modeler some discretion as to the choice of numéraire.[13] The consumer price index (*CPI*) is defined as a weighted sum of the commodity prices (N1), where the weights are the value shares of each commodity in final demand ($comtotsh_c$). The model contains two alternative definitions for the final demand weights; these are either based on total final demands by households or the final demands by all domestic institutionsr – the former is the default. The *CPI* will be used for price normalisation (see below).

The domestic producer price index (*PPI*) is defined as the weighted sum of the commodity prices received by producers on the domestic market (N2), where the weights are

12 Alternative forms of domestic commodity taxes can be specified relatively easily.
13 Other alternatives are available as numéraire, e.g., the exchange rate.

the value shares of each commodity supplied by domestic producers to the domestic market ($vddtotsh_c$). This provides a convenient alternative price normalisation term; if the exchange rate is also fixed it serves to fix the real exchange rate.

Numéraire Block Equations

$$CPI = \sum_c comtotsh_c * PQD_c \qquad \forall c \qquad \text{(N1)}$$

$$PPI = \sum_c vddtotsh_c * PD_c \qquad \forall c \qquad \text{(N2)}$$

<u>Production Block Equations</u>

The output price by activity is defined by the share of commodity outputs produced by each activity, (P1), where, for this case, the weights (*ioqxcqx*) are equal to one where the commodities and activities match and zero otherwise. The weights are derived from the information in the supply (make) matrix.

The production function is a Cobb-Douglas aggregation function over the factors that are demanded by each activity ($FD_{f,a}$), (P2). Each production function has been calibrated for an efficiency parameter (ad_a) and share parameters ($\alpha_{f,a}$).

The first order condition for profit maximisation exploits the properties of Euler's theorem and the relationship between the elasticity of output, i.e., the exponents in the Cobb Douglas function, and factor shares in long-run competitive equilibrium, i.e., the elasticity of output for input *f* in activity *a* is the share of output from activity *a* received by factor *f*. But the 'products' to be distributed among the factors are now defined by reference to the value-added prices rather than activity prices, (P3). Note that the factor-activity specific factor price is the product of the average price for a factor (WF_f) and a set of weights that reflect differences in relatively productivities by activity, ($WFDIST_{f,a}$).

The production block also needs extending to include an equation (P5) for the (commodity) demand for intermediate inputs by commodity ($QINTD_c$). This is defined as the level of activity outputs multiplied by the quantities of inputs used per unit of output, (P4). There is no need to model intermediate input demand from the perspective of the activity.

This consideration is handled by the inclusion of an allowance for intermediate input demand through the price of value added equations and thereby in the first order conditions for profit maximization.

Production Block Equations: Primary Inputs

$$PX_a = \sum_c ioqxcqx_{a,c} * PXC_c \quad (P1)$$

$$QX_a = ad_a \prod_f \left(FD_{f,a}\right)^{\alpha_{f,a}} \quad (P2)$$

$$WF_f * WFDIST_{f,a} * FD_{f,a} = QX_a * PVA_a * \alpha_{f,a} \quad (P3)$$

$$PVA_a = \left(PX_a \left(1 - TXADJ * tx_a\right)\right) - \left(\sum_c PQD_c * ioqintdqx_{c,a}\right) \quad (P4)$$

$$QINTD_c = \sum_a ioqintdqx_{c,a} * QINT_a \quad (P5)$$

The presence of intermediate inputs and production taxes in this system requires the definition of the value-added price (*PVA*), i.e., the amount available per unit of output to pay primary inputs, (P4). The production taxes serve to reduce the share of the revenue per unit of activity output that is to be distributed to factors. The value-added prices (*PVA*) are therefore defined as the activity prices less the per unit production taxes (*tx*), from which are subtracted the weighted sum of consumer commodity prices where the weights used are the input-output technical coefficients, (*ioqintdqx*).

The domestic production of commodities (*QXC*) is defined (P6) as the sum of the per unit outputs of specific commodities by activities ($ioqxcqx_{c,a}$) scaled by the outputs by each activity (*QX*). In this model there is a one-to-one relationship between activities and commodities and therefore the matrix *ioqcxqx* is square and block diagonal, i.e., an identity matrix.[14]

[14] This formulation is chosen because it leaves ‘space’ in the model for the subsequent development of a representation of multi-product activities, e.g., as in the STAGE family of models.

Production Block Equations: Commodity Outputs

$$QXC_c = \sum_a ioqxcqx_{c,a} * QX_a \quad \text{(P6)}$$

Factor Block Equations

The total income received by each factor account (YF_f) is defined as the summation of the earnings of that factor across all activities (F1). But factor incomes that are distributed ($YFDIST_f$) are defined net rather than gross, i.e., after depreciation, where depreciation is defined as a fixed proportion of factor incomes ($deprec_f$).

Factor Block Equations

$$YF_f = \sum_a WF_f * WFDIST_{f,a} * FD_{f,a} \quad \text{(F1)}$$

$$YFDIST_f = \left(YF_f - \left(deprec_f * YF_f\right)\right) \quad \text{(F2)}$$

Household Block Equations

Households acquire income from two source in this model; fixed proportion ($hvash_{h,f}$) of factor incomes after depreciation and from the rest of the world (*howor*). Therefore, household income (*YH*) is defined (H1) as a simple sum of factor incomes remitted/distributed to households plus (net) transfers from the rest of the world that are denominated in foreign currency units.

Household Block Equations

$$YH_h = \sum_f \left(hvash_{h,f} * YFDIST_f\right) + \left(howor_h * ER\right) \quad \text{(H1)}$$

$$HEXP_h = \left(YH_h * \left(1 - \left(TYHADJ * tyh_h\right)\right)\right) * \left(1 - \left(SADJ * shh_h\right)\right). \quad \text{(H2)}$$

$$QCD_{c,h} * PQD_c = \left[PQD_c * qcdconst_{c,h}\right] + beta_{c,h} * \left[HEXP_h - \sum_c PQD_c * qcdconst_{c,h}\right]. \quad \text{(H3)}$$

Household consumption demand is derived in two stages. In the first stage (H2) household consumption expenditures (*HEXP*) are defined as household incomes after the payment of direct taxes and savings, NB: how the saving rate is defined as the proportion of after tax income that is saved; this is important for the calibration of the income tax and savings parameters. The household then seeks to choose quantities of each commodity to maximise the utility realised from its consumption expenditures. Multiple utility functions can be chosen, e.g., Cobb-Douglas, Stone-Geary (LES), AIDS, AdAIDS, nested CES, CDE, etc., but in this instance the default utility functions are assumed to be Stone-Geary (H3). Note, (qcdconst) are the subsistence quantities of commodity *c* consumed by household *h*, the marginal budget shares (*beta*) and then allocated over household consumption expenditures (*HEXP*) less total subsistence expenditure.

Government Tax Block Equations

There are five tax instruments. Each is defined as a simple *ad valorem* rate dependent upon the values of imports, exports, sales, or production and the levels of household income. The ‘tax’ rates are all declared as parameters and then for each tax instrument a scaling variable is declared to facilitate policy experiments.

Government Tax Block Equations

$$MTAX = \sum_c \left(TMADJ * tm_c * PWM_c * ER * QM_c\right) \quad \text{(T1)}$$

$$ETAX = \sum_a \left(TEADJ * te_a * PWE_a * ER * QE_a\right) \quad \text{(T2)}$$

$$STAX = \sum_c \left(TSADJ * ts_c * PQS_c * \left(QINTD_c + QCD_c + QGD_c + QINVD_c\right)\right) \quad \text{(T3)}$$

$$ITAX = \sum_{a} \left(TXADJ * tx_a * PX_a * QX_a \right) \qquad \text{(T4)}$$

$$DTAX = \sum_{h} \left(TYHADJ * tyh_h * YH_h \right) \qquad \text{(T5)}$$

Import duties (*MTAX*) are defined by (T1), where *tm* is the tariff rate and *TMADJ* the scaling variable. Export taxes (*ETAX*) are defined by (T2) where *te* is the export duty rate and *TEADJ* the scaling variable. Sales taxes (*STAX*) are defined by (T3) where *ts* is the sales tax rate and *TSADJ* the scaling variable. Indirect/production taxes (*ITAX*) are defined by (T4) where *tx* is the indirect tax rate and *TXADJ* the scaling variable. And direct (income) taxes (*DTAX*) are defined by (T5) where *tyh* is the household income tax rate and *TYHADJ* the scaling variable.

Government Block Equations

Government income is defined as the sum of government tax revenues (G1), where the tax revenues are treated as expenditures by the accounts paying the taxes and hence are defined in the tax block, and (net) transfers from the rest of the world that are denominated in foreign currency units. Although this approach adds equations it has the arguable advantage of being more transparent, easier to modify and some potential benefits when running policy experiments, e.g., target or minimum or maximum revenues associated with each tax instrument can be defined when running policy experiments.

Government demand for commodities (G2) is assumed to be fixed in (proportionate) real terms (G2), i.e., the volume is fixed, but can be scaled or allowed to vary using an adjustment factor (*QGDADJ*). The precise specification depends upon the choice of closure rule (see below). Thereafter Government consumption expenditure (*EG*) is defined as the sum of the expenditures on commodity consumption (G3). (G2) and (G3) could be collapsed into a single equation but this specification allows some additional options when defining how the government's account is cleared.

Government Block Equations

$$YG = MTAX + ETAX + STAX + ITAX + DTAX + \left(govwor * ER \right). \qquad \text{(G1)}$$

$$QGD_c = qgdconst_c * QGDADJ \quad \text{(G2)}$$

$$EG = \left(\sum_c QGD_c * PQD_c \right) \quad \text{(G3)}$$

Kapital Account Block Equations

Income to the kapital (savings and investment) account comes from household savings, government savings (*KAPGOV*) and the surplus on the capital account of the balance of payments (*KAPWOR*) (K1). Hence total savings are defined as a simple sum across savings by different institutions.

In this model the households are assumed to save fixed proportions (*shh*) of their income after tax, with both the household savings and income tax rates multiplied by adjustment variables (*SADJ* and *TYHADJ*). The savings rate adjustment variable means savings rates can adjust to achieve an exogenously defined level of total savings if an investment driven closure rule is assumed. It may be helpful to assign (K1) simultaneously with (H2) to ensure the correct calibration of the income tax and saving rate parameters. Government savings are calculated as a residual (see the *KAPGOV* equations below): note that the surplus on the capital account (*KAPWOR*) is defined in terms of the foreign currency and therefore the exchange rate appears in this equation (this is a matter of preference).

Kapital Account Block Equations

$$\begin{aligned} TOTSAV = & \sum_h \left(\left(YH_h * \left(1 - \left(TYHADJ * tyh_h \right) \right) \right) * \left(SADJ * shh_h \right) \right) \\ & + \sum_f deprec_f * YF_f \\ & + KAPGOV + \left(KAPWOR * ER \right) \end{aligned} \quad \text{(K1)}$$

$$QINVD_c = IADJ * qinvdconst_c \quad \text{(K2)}$$

$$INVEST = \sum_c \left(PQD_c * QINVD_c \right) \quad \text{(K3)}$$

Investment demand is modelled in a similar way to government demand. Demand for commodities (K2) used in investment is assumed to be in fixed (proportionate) volumes (*qinvdconst*) multiplied by an investment-scaling variable (*IADJ*) that can accommodate changes in the exogenously determined level of investment and/or changes in the availability of funds for investment. The second stage (K3) captures the price effect by identifying the total value of investment (*INVEST*). (K2) and (K3) could be collapsed into a single equation but this specification allows some additional options when defining how the kapital account is closed.

Market Clearing Block Equations

The market clearing, or equilibrium, conditions are straightforward. Factor supplies must equal factor demands (MC1) and (composite) commodity supplies must equal (composite) commodity demands (MC2). It appears that there is no equilibrium condition for the supply of domestic output to the domestic market. In fact, this is achieved indirectly through the commodity output equation (X4), which could have been treated as a market clearing equation.

Market Clearing Block Equations

$$FS_f = \sum_a FD_{f,a} \qquad \text{(MC1)}$$

$$QQ_c = QINTD_c + \sum_h QCD_{c,h} + QGD_c + QINVD_c \;. \qquad \text{(MC2)}$$

$$KAPGOV = YG - EG \;. \qquad \text{(MC3)}$$

$$KAPWOR = \left[\left(\sum_c PWM_c * QM_c\right) - \left(\sum_c PWE_c * QE_c\right)\right] - \left(howor_h * ER\right) - \left(govwor * ER\right) \;. \qquad \text{(MC4)}$$

$$TOTSAV = INVEST + WALRAS \;. \qquad \text{(MC5)}$$

The government account is cleared by defining government savings (*KAPGOV*) as the difference between government income and government expenditure on consumption and transfers; hence government savings are explicitly treated as a residual, (MC3).

The deficit/surplus on the current account, the current account balance, is defined as expenditure on imports less revenue from exports, less net remittances to household ($howor_h$) and less net transfers to the government (*govwor*) all evaluated in terms of the foreign currency (MC4). It is important to bear in mind the difference between the trade balance (the value of imports less the value of exports) and the current account balance: in the base case with the model the current account and trade balances ONLY equate because all other transfers are zero. In experiments, the net transfers from (positive) and to (negative) can be exogenously changed. Note that because *KAPWOR* is defined in terms of the 'foreign' currency it is necessary to convert *KAPWOR* into domestic currency terms when it enters into any other equation, e.g., the total savings equation (*TOTSAV*). And finally, the equilibrium condition for the kapital account includes a slack variable, rather than dropping an equation from the system (MC5).

Table 4 Equation and Variable Counts for the CGE Model

Name	Equation	Number of Equations	Variable	Number of Variables
	EXPORTS BLOCK			
$PEDEF_c$	$PE_c = PWE_c * ER * (1-(TEADJ * te_c)) \quad \forall c \in ce$	c	PE_c	c
CET_c	$QXC_c = at_c\left(\gamma_c * QE_c^{rhot_c} + (1-\gamma_c) * QD_c^{rhot_c}\right)^{\frac{1}{rhot_c}} \quad \forall c \in (cd \cap ce)$	c	QD_c	c
$ESUPPLY_c$	$\frac{QE_c}{QD_c} = \left[\frac{PE_c}{PD_c} * \frac{(1-\gamma_c)}{\gamma_c}\right]^{\frac{1}{(rhot_c - 1)}} \quad \forall c \in (cd \cap ce)$	c	QE_c	c
$CETALT$	$QXC_c = QD_c + QE_c \quad \forall c \in (cd \cap cen)$ or $(cdn \cap ce)$			
	IMPORTS BLOCK			
$PMDEF_c$	$PM_c = PWM_c * ER * (1+(TMADJ * tm_c)) \quad \forall c \in cm$	c	PM_c	c
$ARMINGTON_c$	$QQ_c = ac_c\left(\delta_c * QM_c^{-rhoc_c} + (1-\delta_c) * QD_c^{-rhoc_c}\right)^{-\frac{1}{rhoc_c}} \quad \forall c \in (cx \cap cm)$	c	QQ_c	c
$COSTMIN_c$	$\frac{QM_c}{QD_c} = \left[\frac{PD_c}{PM_c} * \frac{\delta_c}{(1-\delta_c)}\right]^{\frac{1}{(1+rhoc_c)}} \quad \forall c \in (cx \cap cm)$	c	QM_c	c
$ARMALT$	$QQ_c = QD_c + QM_c \quad \forall c \in (cx \cap cmn)$ *or* $\forall c \in (cxn \cap cm)$			

Name	Equation	Number of Equations	Variable	Number of Variables
	COMMODITY PRICE BLOCK			
$PQDEF_c$	$PQD_c = PQS_c * \left(1 + \left(TSADJ * ts_c\right)\right) \quad \forall c \in \left(cd \cup cm\right)$	c	PQD_c	c
$PQSEF_c$	$PQS_c = \dfrac{PD_c * QD_c + PM_c * QM_c}{QQ_c} \quad \forall c \in \left(cd \cup cm\right)$	c	PQS_c	c
$PXCDEF_c$	$PXC_c = \dfrac{PD_c * QD_c + PE_{c \in ce} * QE_{c \in ce}}{QXC_c} \quad \forall c \in cx$	c	PXC_c	c
	NUMERAIRE PRICE BLOCK			
$CPIDEF$	$CPI = \sum_c comtotsh_c * PQ_c \quad \forall c$	1	CPI	1
$PPIDEF$	$PPI = \sum_c vddtotsh_c * PD_c \quad \forall c$	1	PPI	1
	PRODUCTION BLOCK			
$PXDEF_a$	$PX_a = \sum_c ioqxcqx_{a,c} * PXC_c$	a	PX_a	a
$QXPRODFN_a$	$QX_a = ad_a \prod_f \left(FD_{f,a}\right)^{\alpha_{f,a}}$	a	QX_a	a
$QXFOC_{f,a}$	$WF_f * WFDIST_{f,a} * FD_{f,a} = QX_a * PVA_a * \alpha_{f,a}$	$f*a$	$FD_{f,a}$	$f*a$
$QINTEQ_c$	$QINTD_c = \sum_a ioqintdqx_{c,a} * QINT_a$	c	$QINTD_c$	c
$PVADEF_a$	$PVA_a = \left(PX_a \left(1 - TXADJ * tx_a\right)\right) - \left(\sum_c PQD_c * ioqintdqx_{c,a}\right)$	a	PVA_a	a
$COMOUT_c$	$QXC_c = \sum_a ioqxcqx_{a,c} * QX_a$	c	QXC_c	c

Name	Equation	Number of Equations	Variable	Number of Variables
	FACTOR BLOCK			
$YFEQ_f$	$YF_f = \sum_a WF_f * WFDIST_{f,a} * FD_{f,a}$	f	YF_f	f
	HOUSEHOLD BLOCK			
$YHEQ_h$	$YH_h = \left(\sum_f hovash_{h,f} * YF_f \right) + \left(howor_h * ER \right)$	h	YH_h	h
$HEXPEQ_h$	$HEXP_h = \left(YH_h * \left(1 - \left(TYHADJ * tyh_h \right) \right) \right) * \left(1 - \left(SADJ * shh_h \right) \right)$	h	$HEXP$	h
$QCDEQ_{c,h}$	$QCD_{c,h} * PQD_c = \left[PQD_c * qcdconst_{c,h} \right] + beta_{c,h} * \left[HEXP_h - \sum_c PQD_c * qcdconst_{c,h} \right]$	$c*h$	QCD_c	$c*h$

Name	Equation	Number of Equations	Variable	Number of Variables
	GOVERNMENT TAXES BLOCK			
MTAXEQ	$MTAX = \sum_{c}\left(TMADJ * tm_{c} * PWM_{c} * ER * QM_{c}\right)$	1	*MTAX*	1
ETAXEQ	$ETAX = \sum_{a}\left(TEADJ * te_{a} * PWE_{a} * ER * QE_{a}\right)$	1	*ETAX*	1
STAXEQ	$STAX = \sum_{c}\left(TSADJ * ts_{c} * PQS_{c} * \left(QINTD_{c} + QCD_{c} + QGD_{c} + QINVD_{c}\right)\right)$	1	*STAX*	1
ITAXEQ	$ITAX = \sum_{a}\left(TXADJ * tx_{a} * PX_{a} * QX_{a}\right)$	1	*ITAX*	1
DTAXEQ	$DTAX = \sum_{h}\left(TYHADJ * tyh_{h} * YH_{h}\right)$	1	*DTAX*	1
	GOVERNMENT INCOME AND EXPENDITURE BLOCK			
YGEQ	$YG = MTAX + ETAX + STAX + ITAX + DTAX + \left(govwor * ER\right)$	1	*YG*	1
$QGDEQ_c$	$QGD_{c} = qgdconst_{c} * QGDADJ$	*c*	QGD_{c}	*c*
EGEQ	$EG = \left(\sum_{c} QGD_{c} * PQD_{c}\right)$	1	*EG*	1

Name	Equation	Number of Equations	Variable	Number of Variables
	KAPITAL ACCOUNT BLOCK			
TOTSAVEQ	$TOTSAV = \sum_h \left(\left(YH_h * \left(1 - \left(TYHADJ * tyh_h\right)\right)\right) * \left(SADJ * shh_h\right)\right) + KAPGOV + \left(KAPWOR * ER\right)$	1	*TOTSAV*	1
QINVDEQ$_c$	$QINVD_c = IADJ * qinvdconst_c$	*c*	$QINVD_c$	c
INVESTEQ	$INVEST = \sum_c \left(PQD_c * QINVD_c\right)$	1	*INVEST*	1
	MARKET CLEARING BLOCK			
FMEQUIL$_f$	$FS_f = \sum_a FD_{f,a}$	*f*	FS_f	c
QEQUIL$_c$	$QQ_c = QINTD_c + \sum_h QCD_{c,h} + QGD_c + QINVD_c$	*c*	QQ_c	c
KAPGOVEQ	$KAPGOV = YG - EG$	1	*KAPGOV*	1
KAPEQUIL	$KAPWOR = \left[\left(\sum_c PWM_c * QM_c \right) - \left(\sum_c PWE_c * QE_c \right) \right] - \left(howor_h * ER\right) - \left(govwor * ER\right)$	1	*KAPWOR*	1
WALRASEQ	$TOTSAV = INVEST + WALRAS$	1	*WALRAS*	1

Name	Equation	Number of Equations	Variable	Number of Variables
	MODEL CLOSURE			
			$\overline{ER}$ **or** $\overline{KAPWOR}$	1
			$\overline{PWM}$ **and** $\overline{PWE}$	
			$\overline{SADJ}$ **or** $\overline{IADJ}$ **or** $\overline{INVEST}$ **or** $\overline{INVESTSH}$	1
			All **but** two of $\overline{TMADJ}, \overline{TSADJ}, \overline{TEADJ}, \overline{TXADJ}, \overline{TYADJ}, \overline{QGDADJ}, \overline{KAPGOV}, \overline{EG}$	6
			$\overline{WFDIST_{f,a}}, \overline{FS_f}$	3
			$\overline{CPI}$ **or** $\overline{PPI}$	1

4. Market Clearing and Macroeconomic Closure Conditions

In mathematical programming terms the model closure conditions are, at their simplest, a matter of ensuring that the numbers of equations and variables are consistent. However economic theoretic dimensions of model closure rules are more complex, and, as would be expected in the context of an economic model, more important. The essence of model closure rules is that they define important and fundamental differences in perceptions of how an economic system operates (see Sen, 1963; Pyatt, 1987; Kilkenny and Robinson, 1990). The closure rules can be perceived of as operating on two levels; on a general level whereby the closure rules relate to macroeconomic considerations, e.g., is investment expenditure determined by the volume of savings or exogenously, and on a specific level where the closure rules are used to capture particular features of an economic system, e.g., the degree of intersectoral capital mobility.

This model is designed to facilitate changes in model closure with respect to four groups of markets: the foreign exchange market, the kapital (investment-savings) market, the Government's account and the factor markets. The model is set up with a series of simple alternative closures. For the foreign exchange and kapital markets the alternatives are of the either/or kind. For the government account the alternatives are more complex since they allow for policy changes with respect to each of the tax instruments, for adjustments in the volume of government consumption and changes in the balance on the government's budget. More sophisticated and complex configurations can be implemented if appropriate. The basic specification of model assumes full employment and fully flexible factor markets. These assumptions can be relaxed so that factors are sector specific and so that unemployment can be assumed for some factors.

This model allows for the exploration of a wide range of different policy scenarios, although this model is not suitable for real world policy analysis due to limitations imposed by the data and the simple set of behavioural relationships.[15] But the model allows for a range of both general and specific closure rules. The discussion below provides details of the main options available with this formulation of the model by reference to the accounts to which the rules refer.

15 The STAGE and GLOBE models are designed for real world policy analyses.

Foreign Exchange Account Closure

The closure of the rest of the world account can be achieved by fixing either the exchange rate variable (C1a) or the balance on the current account (C1b). In the base specification of the model the exchange rate (*ER*) is fixed, and the current account balance (*KAPWOR*) is the equilibrating variable for the foreign exchange market. Increasingly economies are constrained to operate a flexible exchange rate policy while simultaneously ensuring that the external (current account) balance does not exceed a 'sustainable' deficit. This deficit may be positive or negative. Fixing the exchange rate is appropriate for countries with a fixed exchange rate regime whilst fixing the current account balance is appropriate for countries that face restrictions on the value of the current account balance, e.g., countries following structural adjustment programmes.

It is a common practice to fix a variable at its initial level by using the associated parameter, i.e., ***0, but it is possible to fix the variable to any appropriate value.

$$ER = \overline{ER} \tag{C1a}$$

$$KAPWOR = \overline{KAPWOR} \tag{C1b}$$

$$PWM_c = \overline{PWM_c} \tag{C1c}$$

$$PWE_c = \overline{PWE_c} \tag{C1d}$$

This model adopts the convention of declaring the prices of imported and exported goods as variables. For imported goods it is probably reasonable for the majority of countries to assume that they are price takers on global markets and therefore to fix the price of imports. In this model this amounts to assuming that the import, or *cif*, prices are fixed, (C1c).

Likewise, in this model it is assumed that exporters are price takers on global markets. The *fob* export prices are fixed for all commodities, (C1d).

Kapital Account Closure

The base specification for the kapital (investment_savings) closure is that the total value of investment is determined by the total value of savings, where savings consist of savings from the household, factors (depreciation), government and the foreign exchange account. For the household the base specification assumes that the rate of saving out of after-tax income is fixed at its initial rate. *Ceteris paribus* changes in the household savings rate will alter the total value of investment but the volume of investment also depends upon the price of the investment goods. As specified the model assumes that the proportions of each commodity purchased for investment purposes are fixed in volume terms, via the parameter *qinvdconst*, and hence that the amount of each commodity purchased adjusts equi proportionately so that the total value of savings equals the total value of investment as prices change. This is achieved by fixing the investment-scaling factor *IADJ*, (C2b) If the model is specified so that investment volumes are fixed exogenously then the investment-scaling factor *IADJ* is fixed and the household savings rate is made variable.

Note how the variable *IADJ* determines the ‘level’ of investment volumes and the parameter *qinvdconst* determines the ‘pattern’ of investment volumes. Consequently, doubling *IADJ* will double the volume of investment for each commodity, i.e., *IADJ* imposes an equiproportionate change on the volume of investment for each commodity purchased for investment purposes. If it is proposed to change the pattern of investment volumes it is necessary to change the parameter *qinvdconst*.

However, the *ceteris paribus* assumption ignores the possibility of changes in the other sources of savings, the government and foreign exchange accounts. These can change because of alterations to many of the model parameters, and because of different assumptions about the appropriate closure rules for the foreign exchange market and government account. This highlights the importance of thinking about the model closure rules as a set rather than individually and in isolation.

To ensure that aggregate savings equal aggregate investment, the determinants of either savings or investment must be fixed. This is achieved by fixing either the saving rates for households or the volumes of commodity investment. This involves fixing either the savings rates adjuster (C2a) or the investment volume adjuster (C2b).

Note that fixing the investment volume adjuster (C2b) means that the value of investment expenditure might change due to changes in the prices of investment commodities (*PQD*). Note also that the adjustment in such cases takes place through equiproportionate changes in the savings rates of households despite the fact that there are other sources of savings. The magnitudes of these other savings sources can also be changed through the closure rules (see below).

$$SADJ = \overline{SADJ} \qquad \text{(C2a)}$$

$$IADJ = \overline{IADJ} \qquad \text{(C2b)}$$

$$INVEST = \overline{INVEST} \qquad \text{(C2c)}$$

$$INVESTSH = \overline{INVESTSH} \qquad \text{(C2d)}$$

Fixing savings, and thus deeming the economy to be savings-driven, can be considered a new/neo-classical approach. Closing the economy by fixing investment however makes the model reflect the Keynesian investment-driven assumption for the operation of an economy.

The model includes a variable for the value of investment (*INVEST*), which can also be used to close the kapital account. If *INVEST* is fixed in an investment driven closure, (C2c), then the model will need to adjust the savings rates to maintain equilibrium between the value of savings (*TOTSAV*) and the fixed value of investment. This can only be achieved by changes in the volumes of commodities demanded for investment (*QINVD*) or their prices (*PQD*). But the prices (*PQD*) depend on much more than investment, hence the main adjustment must take place through the volumes of commodities demanded, i.e., *QINVD*, and therefore the volume adjuster (*IADJ*) must be variable, as must the savings rate adjusters (*SADJ*).

Alternatively the share of investment expenditure in the total value of domestic final demand (*INVESTSH*) can be fixed, (C2d), which means that the total value of investment is fixed by reference to the value of total final demand, but otherwise the adjustment mechanisms follow the same processes as for fixing *INVEST* equal to some level.

Government Account Closure

The specification of the closure for the government account in the base model presumes that all tax rates are fixed at their initial rates and that government expenditure volumes are fixed, and therefore that government saving, the internal balance, is a residual. However, it is a common government policy objective to identify the tax rates, or rate, that are consistent with some pre-defined level of government saving (positive or negative). This requires that the value of government saving is fixed (in the base model the default fixed value is the initial value), and that at least one other variable under the government's control is unfixed. The model is specified so that there are ten variables under the government's control that can be fixed or unfixed: six tax instruments, government consumption volumes, government expenditure, government share of absorption and government saving.

The specification for the tax rates and the volume of government consumption each use the method of defining a scaling variable and parameters that define the relative patterns of the tax rates and consumption. As with investment scaling factor, these scaling factors are used to adjust the level of the tax burden or consumption while changes in the tax rate and consumption parameters alter the relative patterns of taxes and consumption.

For instance, assume that the objective is to find a uniform tariff rate that is equivalent to the current non-uniform tariff rate in terms of its impact upon the government account, i.e., that leaves the level of government saving unchanged. The variable *KAPGOV* would be fixed at its initial level (*KAPGOV0*) and the variable *TMADJ* would be unfixed. Then the parameter *tm* would be changed so that the relative tariff rate on all imported commodities was made identical, say 0.1.[16] When the model is solved the effective uniform tariff rate, which satisfies the constraints on the government account, would be 0.1 times *TMADJ*. Note however that the patterns of production, consumption and trade are likely to have changed, as would the incomes to the household and government, and the level of GDP. The economy may be better off or worse off.

The range of options the closure rules for the government account are slightly trickier because they are important components of the model that are used to investigate fiscal policy considerations. The base specification uses the assumption that government savings are a

16 If *tm* is fixed for all *c* this can cause problems when there are commodities that are not imported. Hence *tm* should be fixed for all *cm*, while *tm* for all *cmn* equals zero.

residual; when the determinants of government income and expenditure are ‘fixed’, government savings must be free to adjust.

$$TMADJ = \overline{TMADJ} \quad (C3a)$$

$$TEADJ = \overline{TEADJ} \quad (C3b)$$

$$TSADJ = \overline{TSADJ} \quad (C3c)$$

$$TXADJ = \overline{TXADJ} \quad (C3d)$$

$$TYHADJ = \overline{TYHADJ} \quad (C3e)$$

$$QGDADJ = \overline{QGDADJ} \quad (C3f)$$

$$VGDSH = \overline{VGDSH} \quad (C3g)$$

$$EG = \overline{EG} \quad (C3h)$$

$$KAPGOV = \overline{KAPGOV} \quad (C3i)$$

Thus, in the base specification all the tax rates are fixed by declaring the tax rates as parameters and then fixing all the tax rate scaling factors (C3a – C3e). Consequently, changes in tax revenue to the government are consequences of changes in the other variables that enter into the tax income equations (T1 to T6).

In the base specification government expenditure is controlled by fixing the volumes of commodity demand (*QGD*) through the government demand adjuster (*QGDADJ*), (C3f), or the share of government expenditure in the total value of domestic final demand can be fixed, (C3g). Either specification ensures that all the parameters that the government can/does control are fixed and consequently that the only determinants of government income and expenditure that are free to vary are those that the government does not directly control. Hence the equilibrating condition is that government savings, the internal balance, is not fixed. Alternatively, rather than controlling the volume of government expenditure, the value of government expenditure could be fixed, (C3h).

If however the model requires government savings to be fixed (C3i), then either government income or expenditure must be free to adjust. Such a condition might reasonably be expected in many circumstances, e.g., the government might define an acceptable level of borrowing, or such a condition might be imposed externally.

In its simplest form this can be achieved by allowing one of the previously fixed adjusters (C3a to C3h) to vary. Thus, if the sales tax adjuster (*TSADJ*) is made variable then the sales tax rates will be varied equiproportionately so as to satisfy the internal balance condition. More complex experiments might result from the imposition of multiple conditions, e.g., a halving of import duty rates coupled with a reduction in government deficit, in which case the variables *TMADJ* and *KAPGOV* would also require resetting. But these conditions might create a model that is infeasible, e.g., due to insufficient flexibility through the sales tax mechanism, or unrealistically high rates of sales taxes. In such circumstances it may be necessary to allow adjustments in multiple tax adjusters. One method then would be to fix the tax adjusters to move in parallel with each other; this may involve the addition of 'behavioural' terms to the model that facilitate the implementation of policy experiments. These additional 'behavioural' terms often obscure the essentially simple mechanisms that operate in this class of model.

However, if the adjustments only take place through the tax rate scaling factors the relative tax rates will be fixed. To change relative tax rates, it is necessary to change the relevant tax parameters. Typically, such changes would be implemented in policy experiment files rather than within the closure section of the model; versions of this class of model with more complex tax rate setting equations do exist, but the basic principles are unchanged, and the equations make the mechanisms at play less transparent.

Numéraire

The model specification allows for a choice of two price normalisation equations, the consumer price index (C4a) and a producer price index, (C4b). A *numéraire* is needed to serve as a base since the model is homogenous of degree zero in prices and hence only defines relative prices.

$$CPI = \overline{CPI} \tag{C4a}$$

$$PPI = \overline{PPI} \tag{C4b}$$

Factor Market Clearing

The factor market clearing rules are more difficult to implement than many of the other closure options. Hence the discussion below proceeds in three stages; the first stage sets up a basic specification whereby all factors are deemed perfectly mobile, the second stage introduces a more general specification whereby factors can be made activity specific and allowance can be made for unemployed factors, while the third stage introduces the idea that factor market restrictions may arise from activity specific characteristics, rather than the factor inspired restrictions considered in the second stage.

Full Factor Mobility and Employment Clearing

This factor market closure requires that the total supply of and total demand for factors equate. The total supplies of each factor are determined exogenously and hence (C5a) defines the first set of factor market closure conditions. The demands for factor *f* by activity *a* and the wage rates for factors are determined endogenously. But the model specification includes the assumption that the wage rates for factors are averages, by allowing for the possibility that the payments to notionally identical factors might vary across activities through the variable that captures the 'sectoral proportions for factor prices', i.e., activity specific differences in factor productivity. These proportions are assumed to be a consequence of the use made by activities of factors, rather than of the factors themselves, and are therefore assumed fixed, (C5b).

Finally bounds are placed upon the average factor prices, (C5c), so that meaningful results are produced.

$$FS_f = \overline{FS}_f \tag{C5a}$$

$$WFDIST_{f,a} = \overline{WFDIST_{f,a}} \tag{C5b}$$

$$\begin{aligned} &\text{Min } WF_f = 0 \\ &\text{Max } WF_f = +\text{infinity} \end{aligned} \tag{C5c}$$

Alternative Factor Market Clearing Conditions

More general factor market clearing conditions wherein factor immobility and/or factor unemployment are assumed can be achieved by determining which variables referring to factors are treated as variables and which variables are treated as parameters. If factor market clearing rules are changed it is important to be careful to preserve the equation and variable counts when relaxing conditions, i.e., converting parameters into variables, and imposing conditions, i.e., converting variables into parameters, while preserving the economic logic of the model.

A convenient/simple way to proceed is to define a block of conditions for each factor, or subset of factors. For this model this amounts to defining the following possible equations (C5d), where *fact* indicates the specific factor. The block of equations in (C5d) includes all the variables that were declared for the model with reference to factors. The choice of which equations are binding, and which are not imposed, i.e., which variables are fixed (parameters) and which are variable, will determine the factor market clearing conditions.

As can be seen the first four equations in the block (C5d) are the same as those in the ‘Full Factor Mobility and Employment Clearing’; hence ensuring that these four equations are operating for each of the factors is a longhand method for imposing ‘Full Factor Mobility and Employment Clearing’. Assume that this set of conditions represents the starting point, i.e., the first four equations are binding, and hence the last six equations are not imposed.

$$FS_{fact} = \overline{FS_{fact}}$$
$$WFDIST_{fact,a} = \overline{WFDIST_{fact,a}}$$
$$\text{Min } WF_{fact} = -\inf$$
$$\text{Max } WF_{fact} = +\inf$$
$$FD_{fact,a} = \overline{FD_{fact,a}}$$
$$WF_{fact} = \overline{WF_{fact}}$$
$$WFDIST_{fact,a} = -\inf$$
$$WFDIST_{fact,a} = +\inf$$
$$\text{Min } FS_{fact} = -\inf$$
$$\text{Max } FS_{fact} = +\inf$$
(C5d)

Factor Immobility Clearing

Assume that it is objective is to impose short run clearing on the model, whereby a factor is assumed to be activity specific, and hence there is no inter sectoral factor mobility. Typically, this would involve making capital activity specific and immobile, although it can be applied to any factor. This requires imposing the condition that factor demands are activity specific, i.e., the condition (C5e) must be imposed. But the returns to this factor in different uses (activities) must now be allowed to vary, i.e., the condition (C5f) must now be relaxed. It is 'good' practice to make sure that the variables *WFDIST* are flexible by 'switching' on the conditions $WFDIST_{fact,a} = -\inf / +\inf$. This ensures that '*a*' variables are fixed and '*a*' variables are flexed.

$$FD_{fact,a} = \overline{FD}_{fact,a} \tag{C5e}$$

$$WFDIST_{fact,a} = \overline{WFDIST}_{fact,a} \tag{C5f}$$

$$FS_{fact} = \overline{FS}_{fact} \tag{C5g}$$

$$WF_{fact} = \overline{WF}_{fact} \tag{C5h}$$

The number of imposed conditions is equal to the number of relaxed conditions, which suggests that the model will still be consistent. But the condition fixing the total supply of the factor is redundant since if factor demands by activity are fixed the total factor supply cannot vary. Hence the condition (C5g) is redundant and should be relaxed, and one other constraint imposed. This can be achieved by fixing the average price for the factor, i.e., (C5h), which means that changes in activity specific returns to the factor will be reported as changes in the variables $WFDIST_{fact,a}$.[17]

Surplus Labour Clearing

Consider the case where there is 'surplus' unskilled labour; this is often described as unemployment although that is a somewhat imprecise description. In a simple representation of surplus unskilled labour, the supply of the unskilled labour is unconstrained and there is an infinitely elastic supply of unskilled labour at a constant price: hence the wage rate for unskilled labour is constant.

Start from the closure conditions for full factor mobility and employments and then assume that there is surplus labour of one or more factors in the economy; typically, this would be one type or another of unskilled labour. If the supply of the surplus factor is perfectly elastic, then activities can employ any amount of that factor at a fixed price. This requires imposing the condition (C5i) and relaxing the assumption that the total supply of the factor is fixed at the base level, i.e., relaxing (C5j).

$$WF_{fact} = \overline{WF}_{fact} \qquad \text{(C5i)}$$

$$FS_{fact} = \overline{FS}_{fact} \qquad \text{(C5j)}$$

$$\begin{aligned} &\text{Min } FS_{fact} = -\text{inf} \\ &\text{Max } FS_{fact} = +\text{inf} \end{aligned} \qquad \text{(C5k)}$$

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[17] An alternative approach is to define returns to the factor relative to a reference activity, by fixing one *WFDIST*. This can present difficulties if a factor is not used, or little used, by the chosen activity.

It is useful however to impose some restrictions on the total supply of the factor that is unemployed. Hence the conditions (C5k) can be imposed.[18]

Activity Inspired Restrictions on Factor Market Clearing

There are circumstances where factor use by an activity might be restricted because of activity specific characteristics. For instance, it might be assumed that the volume of production by an activity might be predetermined, e.g., the volume of mineral resources that can be extracted in a year may be fixed and/or there might be an exogenously fixed restriction upon the rate of extraction of a mineral commodity. In such cases the objective may be to fix the quantities of all factors used by an activity, rather than to fix the amounts of a factor used by all activities. This is clearly a variation on the factor market clearing conditions for making a factor activity specific.

$$FD_{f,activ} = \overline{FD}_{f,activ} \tag{C5l}$$

$$WFDIST_{f,activ} = \overline{WFDIST}_{f,activ} \tag{C5m}$$

If all factors used by an activity are fixed, this requires imposing the conditions (C5l), where *activ* refers to the activity of concern. But the returns to these factors in this activity must now be allowed to vary, i.e., the conditions (C5m) must now be relaxed. In this case the condition fixing the total supply of the factor is not redundant since only the factor demands by *activ* are fixed and the factor supplies to be allocated across other activities are the total supplies unaccounted for by *activ*.[19]

Such conditions can be imposed by extending the blocks of equations for each factor in the factor market closure section. However, it is often easier to manage the model by gathering factor market conditions that are inspired by activity characteristics after the factor

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18 If the total demand for the unemployed factor increases unrealistically in the policy simulations, then it is possible to place an upper bound of the supply of the factor and then allow the wage rate from that factor to vary.

19 In more complex models the nesting of the production structure may allow for substitution between total primary and total intermediate inputs, e.g., with CES technologies. In such models simply fixing the quantities of primary inputs is not enough to ensure that the output of an activity will be fixed; one solution is to choose Leontief technologies rather than CES technologies for the specified activity.

inspired equations. In this context it is useful to note that when working in GAMS that the last condition imposed, in terms of the order of the code, is binding and supersedes previous conditions.

Appendix 1

Table A1 Model Commodity and Activity Accounts and Mapping to GTAP – ModO5

Commodities			Activities	
Model	**GTAP**	**Description**	**Model**	**GTAP**
cagric	ccpdr	Paddy rice	aagric	aapdr
	ccwht	Wheat		aawht
	ccgro	Cereal grains nec		aagro
	ccv_f	Vegetables fruit nuts		aav_f
	ccosd	Oil seeds		aaosd
	ccc_b	Sugar cane sugar beet		aac_b
	ccpfb	Plant based fibers		aapfb
	ccocr	Crops nec		aaocr
	ccctl	Bovine cattle sheep and goats horses		aactl
	ccoap	Animal products nec		aaoap
	ccrmk	Raw milk		aarmk
	ccwol	Wool silk worm cocoons		aawol
	ccfrs	Forestry		aafrs
	ccfsh	Fishing		aafsh
cnres	cccol	Coal	anres	aacol
	ccoil	Oil		aaoil
	ccgas	Gas		aagas
	ccomn	Minerals nec		aaomn
cmanu	cccmt	Bovine cattle sheep and goat horse meat prods	amanu	aacmt
	ccomt	Meat products nec		aaomt
	ccvol	Vegetable oils and fats		aavol
	ccmil	Dairy products		aamil
	ccpcr	Processed rice		aapcr
	ccsgr	Sugar		aasgr
	ccofd	Food products nec		aaofd
	ccb_t	Beverages and tobacco products		aab_t
	cctex	Textiles		aatex
	ccwap	Wearing apparel		aawap
	cclea	Leather products		aalea
	cclum	Wood products		aalum
	ccppp	Paper products publishing		aappp
	ccp_c	Petroleum coal products		aap_c
	cccrp	Chemical rubber plastic products		aacrp
	ccnmm	Mineral products nec		aanmm
	cci_s	Ferrous metals		aai_s
	ccnfm	Metals nec		aanfm
	ccfmp	Metal products		aafmp
	ccmvh	Motor vehicles and parts		aamvh
	ccotn	Transport equipment nec		aaotn
	ccele	Electronic equipment		aaele
	ccome	Machinery and equipment nec		aaome
	ccomf	Manufactures nec		aaomf
	ccely	Electricity		aaely
	ccgdt	Gas manufacture distribution		aagdt
	ccwtr	Water		aawtr
	cccns	Construction		aacns
cserv	cctrd	Trade	serv	aatrd
	ccotp	Transport nec		aaotp
	ccwtp	Sea transport		aawtp
	ccatp	Air transport		aaatp
	cccmn	Communication		aacmn
	ccofi	Financial services nec		aaofi
	ccisr	Insurance		aaisr
	ccobs	Business services nec		aaobs
	ccros	Recreation and other services		aaros
	ccosg	PubAdmin Defence Health Educat		aaosg

ccdwe	Dwellings	aadwe

Table A2 Model Factor Accounts and Mapping – ModO5

Model Factor	GTAP Factor	Description	Model Factor Tax	GTAP Factor Tax
fland	fLand	Land	tfland	tffLand
	fNatlRes	Natural Resources		tffNatlRes
fUskil	fUnSkLab	Unskilled labour	tfUskil	tffUnSkLab
fskil	fSkLab	Skilled labour	tfskil	tffSkLab
fcap	fCapital	Capital	tfcap	tffCapital

Table A3 Other Model Accounts

Other Accounts		Other Accounts	
imptax	Import duties	govt	Government
exptax	Export duties	hous	Household
saltax	Sales taxes	i_s	Investment-savings
vattax	Valeu Added taxes	row	Rest of the world
prodtax	Production Taxes	total	Total
dirtax	Direct Taxes		

Appendix 2 Production System for *smod_t2*

Introduction

The production system in *smod_t* can be changed without needing change any of the other behavioural relationships and in such ways as to include production systems that are ‘state-of-the-art’ with respect to primary inputs. The examples explore production systems based on nested Constant Elasticity of Substitution functions that are ubiquitous in CGE models. The first alternative developed, *smod_t2*, relaxes two critical assumptions in the *smod_t* model: the assumption that the elasticity of substitution between primary inputs is one and the assumption that there are no substitution possibilities between aggregate value added and aggregate intermediate inputs. Such two level production systems were common in the early 2000 and was still found in many models, e.g., the GTAP model, IFPRI standard model, etc., in the first half of the 2020s. The second alternative developed, *smod_t3*, relaxes a critical assumption in the *smod_t2* model: the assumption that the elasticity of substitution between all primary inputs is the same. This is achieved by adding an additional nest to create a three level production system.

The use of nested CES functions produces CGE models that, arguably, are more realistic. Importantly, it has been demonstrated (Perroni and Rutherford, 1995) that regular nested CES functions have the properties of flexible function forms, e.g., translog functions, and are ‘well behaved’, i.e., do not produce ‘odd’ results, while being easy to calibrate and interpret. Nested functions are a staple of CGE models and central to most single country and global CGE models and critical to models used to analyse energy, climate change, water, etc., policies. The three level nesting system developed for *smod_t3* can be extended to produce a generalised nesting system with *n* levels of nested CES functions that can encompass not only primary input substitution possibilities but also substitution possibilities between primary and intermediate inputs required for, *inter alia*, energy and climate change analyses. The extensions can be implemented through appropriate set declarations and minimal changes to the core model equations.

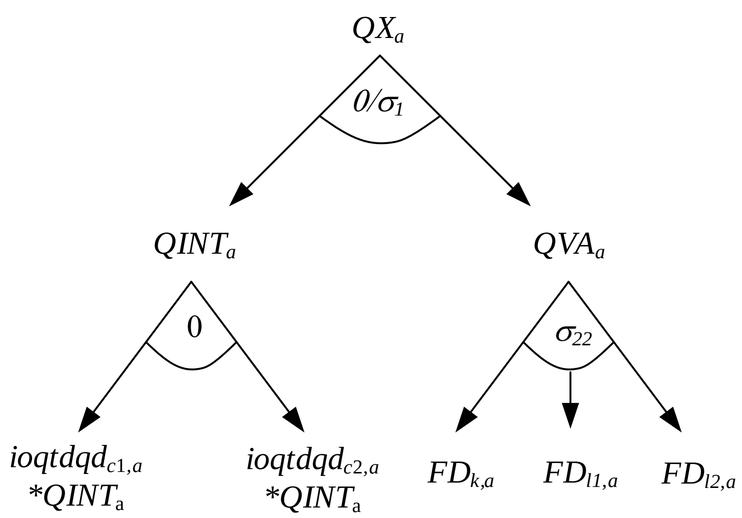
While nested CES functions may initially appear to be a solution to many CGE model issues they are not without costs. Specifically, for each additional nest it is necessary to define substitution elasticities for which there is limited, or no, empirical evidence.

Two Level Production System (*smod_t2*)

The two level production developed here was common in the early 2000 and was still found in many models, e.g., the GTAP model, IFPRI standard model, STAGE_1 model, etc., in the first half of the 2020s.

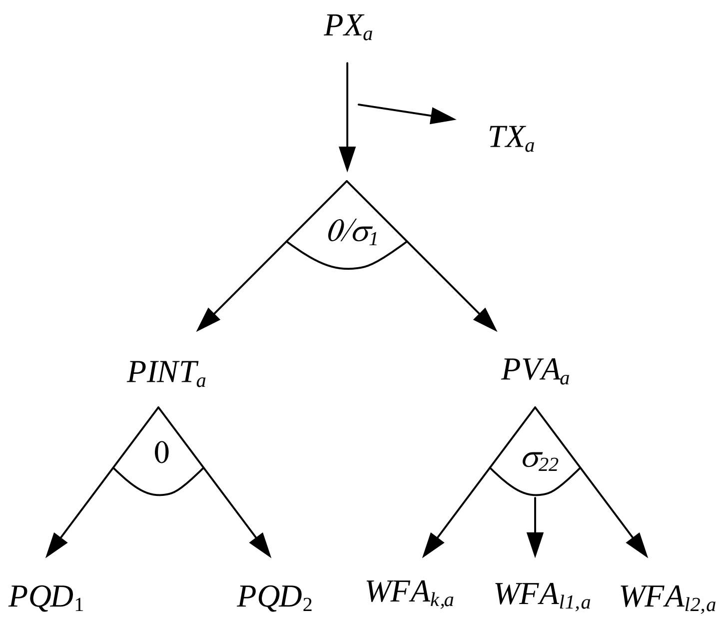
In the *smod_t2* variant, production relationships by activities are defined by a two levels of nested Constant Elasticity of Substitution (CES) production functions. The two level production nest, in quantity terms, is illustrated in Figure A2.1. For illustration purposes only, two intermediate inputs and three primary inputs ($FD_{k,a}$, $FD_{l1,a}$ and $FD_{l2,a}$) – one type of capital and 2 types of labour - are identified. Activity output is a CES or Leontief aggregate of the quantities of aggregate intermediate inputs (*QINT*) and value added (*QVA*): σ in the arc indicates a CES function, while 0 indicates Leontief aggregation. Aggregate intermediate inputs are a Leontief aggregate of the (individual) intermediate inputs and aggregate value added is a CES aggregate of the quantities of primary inputs demanded by each activity (*FD*). The allocation of the finite supplies of factors (*FS*) between competing activities depends upon relative factor prices via first order conditions for optima.

Figure A2.1 Two Level Production System: Quantities



The price relations for the production system are illustrated in Figure A2.2. Note how the prices paid for intermediate inputs (*PQD*) are the same as paid for final demands, i.e., the ‘law’ of one price holds across all domestic demand. Note also that factor prices are factor and activity specific ($WFA_{f,a}$); in the model $WFA_{f,a} = WF_f * WDIST_{f,a}$, i.e., the labels used in Figure A2.2 are 'abbreviations'.

Figure A2.2 Two Level Production System: Prices



Production Block

The supply prices of domestically produced commodities are determined by purchaser prices of those commodities on the domestic and international markets. Adopting the assumption that domestic activities produce commodities in fixed proportions (*ioqxacqx*), the proportions provide a mapping (P2.1) between the supply prices of commodities (*PXC*) and the (weighted) average activity prices (*PX*).[20]

In this model a two-stage production process is adopted, with the top level as a CES or Leontief function. If a CES is imposed for an activity the value of activity output can be expressed as the volume share weighted sums of the expenditures on inputs after allowing for the production taxes (*TX*), which are assumed to be applied *ad valorem* (P2.2). This requires the definition of aggregate prices for intermediates (*PINT*); these are defined as the intermediate input-output coefficient weighted sum of the prices of intermediate inputs (P2.3),

20 In the special case of each activity producing only one commodity **and** each commodity only being produced by a single activity, which is the case in the reduced form model reported in Dervis *et al.*, (1982), then the aggregation weights *ioqxacqx* correspond to an identity matrix. This is also the case with the *smod* suite.

where $ioqtdqd_{c,a}$ are the intermediate input-output coefficients where the output is the aggregate intermediate input (*QINT*).

With CES technology the output by an activity, (*QX*) is determined by the aggregate quantities of factors used (*QVA*), i.e., aggregate value added, and aggregate intermediates used (*QINT*), where δ_a^x is the share parameter, $rhoc_a^x$ is the substitution parameter and AD_a^x is the efficiency variable (P2.5). Note how the efficiency/shift factor is defined as a variable and an adjustment mechanism is provided (P2.4), where *adxb* is the base values, *dabadx* is an absolute change in the base value, *ADXADJ* is an equiproportionate (multiplicative) adjustment factor, *DADX* is an additive adjustment factor and *adx*01 is a vector of zeros and non-zeros used to scale the additive adjustment factor. The operation of this type of adjustment equation has been explored in the *smod_t* model. The associated the first order conditions defining the optimum ratios of value added to intermediate inputs can be expressed in terms of the relative prices of value added (*PVA*) and intermediate inputs (*PINT*), see (P2.6).

Production Block Equations: Top Level

$$PX_a = \sum_c ioqxacqx_{a,c} * PXC_c \quad . \qquad \text{(P2.1)}$$

$$PX_a * (1 - TX_a) * QX_a = (PVA_a * QVA_a) + (PINT_a * QINT_a) \quad . \qquad \text{(P2.2)}$$

$$PINT_a = \sum_c (ioqtdqd_{c,a} * PQD)_c \qquad \text{(P2.3)}$$

$$ADX_a = \left[(adxb_a + dabadx_a) * ADXADJ \right] + (DADX * adx01_a) \qquad \text{(P2.4)}$$

$$QX_a = AD_a^x \left(\delta_a^x QVA_a^{-rhoc_a^x} + (1 - \delta_a^x) QINT_a^{-rhoc_a^x} \right)^{-\frac{1}{rhoc_a^x}} \qquad \forall aqx_a \quad . \qquad \text{(P2.5)}$$

$$\frac{QVA_a}{QINT_a} = \left[\frac{PINT_a}{PVA_a} * \frac{\delta_a^x}{(1 - \delta_a^x)} \right]^{\frac{1}{(1+rhoc_a^x)}} \qquad \forall aqx_a \quad . \qquad \text{(P2.6)}$$

$$QVA_a = ioqvaqx_a * QX_a \qquad \forall aqxn_a \qquad \text{(P2.7a)}$$

$$QINT_a = ioqintqx_a * QX_a \qquad \forall aqx_a \tag{P2.7b}$$

Because there are only two arguments for this CES function the simple two input version can be used.

With Leontief technology at the top level the aggregate quantities of factors used (*QVA*), i.e., aggregate value added, and intermediates used (*QINT*), are determined by simple aggregation functions, (P2.7a) and (P2.7b), where *ioqvaqx* and *ioqintqx* are the (fixed) volume shares of *QVA* and *QINT* (respectively) in *QX*, i.e., the percentage changes in *QX*, *QVA* and *QINT* will be identical. The choice of top level aggregation function is controlled by the membership of the set *aqx*, with the membership of *aqxn* being the complement of *aqx*.

Production Block Equations: Second Level

$$ADVA_a = \left[\left(advab_a + dabadva_a\right) * ADVAADJ\right] + \left(DADVA * adva01_a\right) \tag{P2.8}$$

$$QVA_a = AD_a^{va} * \left[\sum_{f\$\delta_{f,a}^{va}} \delta_{f,a}^{va} * ADFD_{f,a} * FD_{f,a}^{-\rho_a^{va}}\right]^{-1/\rho_a^{va}} \tag{P2.9}$$

$$\begin{aligned}
& WF_f * WFDIST_{f,a} * \left(1 + TF_{f,a}\right) \\
& = PVA_a * AD_a^{va} * \left[\sum_{f\$\delta_{f,a}^{va}} \delta_{f,a}^{va} * ADFD_{f,a} * FD_{f,a}^{-\rho_a^{va}}\right]^{-\left(\frac{1+\rho_a^{va}}{\rho_a^{va}}\right)} * \delta_{f,a}^{va} * FD_{f,a}^{\left(-\rho_a^{va}-1\right)} \\
& = PVA_a * QVA_a * AD_a^{va} * \left[\sum_{f\$\delta_{f,a}^{va}} \delta_{f,a}^{va} * ADFD_{f,a} * FD_{f,a}^{-\rho_a^{va}}\right]^{-1} \\
& \qquad * \delta_{f,a}^{va} * ADFD_{f,a}^{-\rho_a^{va}} * \delta_{f,a}^{va} * FD_{f,a}^{\left(-\rho_a^{va}-1\right)} .
\end{aligned} \tag{P2.10}$$

$$QINTD_c = \sum_a ioqtdqd_{c,a} * QINT_a \tag{P2.11}$$

There are two arms to the second level production nest. For aggregate value added (*QVA*) the production function is a multi-factor CES function (P2.9) where δ_a^{va} is the share parameter, $rhoc_a^{va}$ is the substitution parameter and AD_a^{va} is the efficiency factor. Because there can be more than two inputs to *QVA* the multiple argument variant of the CES function must be used. The associated first order conditions for profit maximisation (P2.10) determine the wage rate of factors (*WF*), where the ratio of factor payments to factor *f* from activity *a* (*WFDIST*) are included to allow for non-homogenous factors, and is derived directly from the first order condition for profit maximisation as equalities between the wage rates for each factor in each activity and the values of the marginal products of those factors in each activity,[21] Again the efficiency/shift factor is defined as a variable with an adjustment mechanism (P2.8), where *advab* is the base values, *dabadva* is an absolute change in the base value, *ADVAADJ* is an equiproportionate (multiplicative) adjustment factor, *DADVA* is an additive adjustment factor and *adva*01 is a vector of zeros and non-zeros used to scale the additive adjustment factor.

The assumption of a two-stage production nest with Constant Elasticity of Substitution between aggregate intermediate input demand and aggregate value added and Leontief technology on intermediate inputs means that intermediate commodity demand (*QINTD*) is defined (P2.11) as the product of the fixed (Leontief) input coefficients of demand for commodity *c* by activity *a* (*ioqtdqd*), multiplied by the quantity of activity intermediate input (*QINT*). Note how, in contrast to the *smod_t* model the input coefficients of demand for commodity *c* by activity *a* are defined relative to the intermediate input aggregates (*QINT*) rather than the activities outputs (*QX*). This reflects the fact that the ratios between *QX* and *QINT* can change due the presence of aggregate substitution possibilities in the CES case.

Production Block Equations: Commodity Outputs

$$QXC_c = \sum_a ioqxcqx_{c,a} * QX_a \qquad \text{(P2.12)}$$

21 The formulation in top line of (P2.10) implies that both the activity outputs (*QX*) and factor demands are solved simultaneously through the profit maximisation process. However, the formulation in the second line is more flexible since, *inter alia*, it allows the possibility of production rationing, i.e., activity outputs (*QX*) were fixed, but there was still cost minimisation. Thanks are due to Sherman Robinson for the explanation as to the practical distinction between these alternative, but mathematically equivalent, formulations.

The domestic production of commodities (*QXC*) is defined (P2.12) as the sum of the per unit outputs of specific commodities by activities ($ioqxcqx_{c,a}$) scaled by the outputs by each activity (*QX*). This is unchanged from the *smod_t* model. In this model there is a one-to-one relationship between activities and commodities and therefore the matrix *ioqcxqx* is square and block diagonal, i.e., an identity matrix.[22]

Calibration

Define the elasticity of substitution for a CES function as σ, then if the elasticity values are given exogenously the elasticity parameters (with the super script *va* to indicate value added and the subscript *a* to indicate activity *a*) are

$$\rho_a^{va} = \frac{1}{\sigma_a^{va}} - 1 \qquad \text{(A2.1)}$$

Assuming the transaction data report an equilibrium state then the 1st order condition holds for the prices and quantities for the recorded transactions, is

$$WF_f * WFDIST0_{f,a} = PVA0_a * QVA0_a \left(\sum \delta_{f,a}^{va} * FD0_{f,a}^{\left(-\rho_a^{va}\right)} \right)^{-1} * \delta_{f,a}^{va} * FD0_{f,a}^{\left(-\rho_a^{va}-1\right)}$$

(A2.2)

where WF_f is the average price of factor *f* and $WFDIST_{f,a}$ is the adjustment for the price of factor *f* in activity *a*, PVA_a and QVA_a are the price and quantity of value added in activity *a*, $\delta_{f,a}^{va}$ is the share parameter of factor *f* in activity *a*, and $FD_{f,a}$ is the quantity of factor *f* used/demanded in activity *a*. The zeros represent initial values. Solving (A2.2) for δ produces

$$\delta_{f,a}^{va} = \frac{WF_f * WFDIST0_{f,a} * FD0_{f,a}^{\left(1+\rho_a^{va}\right)}}{\sum_f WF_f * WFDIST0_{f,a} * FD0_{f,a}^{\left(1+\rho_a^{va}\right)}} \qquad \text{(A2.3)}$$

which since the initial quantities and prices can be derived from the transaction values and the price normalisation rules means that $\delta_{f,a}^{va}$ can be computed because all the RHS terms are known. Given the initial quantities and prices, ρ and δ the function can be solved for the technology parameter $adva_a$, i.e.,

22 This formulation is chosen because it leaves 'space' in the model for the subsequent development of a representation of multi-product activities, e.g., as in the STAGE family of models.

$$adva_a = \frac{QVA0_a}{\left(\sum_f \delta^{va}_{f,a} * FD0_{f,a}^{\left(-\rho^{va}_a\right)}\right)^{\left(-1/\rho^{va}_a\right)}} \qquad \text{(A2.4)}$$

This demonstrates how the CES function is calibrated by working back from the functional form, transaction values and the elasticity parameter when assuming the transaction values represent an equilibrium.

The formulae used depend on whether the two or *n* argument CES equations are used.

The equations for the two argument case, used in this model for the top level of the nest, are

```
rhocx(a)     = (1/ELASTX(a,"sigmax")) - 1 ;
deltax(a)$(QINT0(a))
                    = (PVA0(a)*QVA0(a)**(1+rhocx(a)) )
                         /(PVA0(a)*QVA0(a)**(1+rhocx(a))
                             + PINT0(a)*QINT0(a)**(1+rhocx(a))) ;
ADX0(a)$deltax(a)
         = QX0(a)/(deltax(a)*QVA0(a)**(-rhocx(a))
                  +(1-deltax(a))*QINT0(a)**(-rhocx(a)))**(-1/rhocx(a)) ;
```

Note how all the RHS terms for *deltax* (share parameters), except for *rhocx*, can be derived from the transaction values, while the shift parameters (*ADX*0) require the *deltax* values. Thus before calibrating the share and shift parameters the initial values for the variables need to be derived. The equations of the initial values of the parameters used are

```
QINT0(a)$SUM(c,SAM(c,a)) = SUM(c,SAM(c,a)/PQD0(c)) ;
QVA0(a)                 = SUM(f,SAM(f,a)) ;
PINT0(a)$QINT0(a)       = SUM(c,(SAM(c,a)/PQD0(c)/QINT0(a))*PQD0(c));
PVA0(a)$QVA0(a)         = SUM(f,SAM(f,a))/QVA0(a) ;
```

Note how the equations require the prior assignment of the purchaser prices (*PQD0*) and that aggregate value added (*QVA0*) is expressed in value units and that the price of aggregate value added (*PVA0*) is one but the price of aggregate intermediate inputs is not one (unless there are no differences between basic and purchaser prices of commodities). Initially this may appear odd, but it needs to be remembered that the system is based on changes in relative prices and NOT absolute prices.

The alternative top level assumption is that there are no substitution possibilities between aggregate value added and intermediate inputs. To allow for this option it is necessary to calibrate the Leontief alternative as

```
ioqintqx(a)$QX0(a)    = QINT0(a)/QX0(a)  ;
ioqvaqx(a)$QX0(a)     = QVA0(a)/QX0(a)  ;
```

The equations for the *n*-argument case, used in this model for the second level of the nest, are

```
rhocva(a)    = (1/ELASTX(a,"sigmava")) - 1 ;
deltava(f,a)$SAM(f,a)
             = (WFDIST0(f,a)*WF0(f)*(FD0(f,a))**(1+rhocva(a)))
               /SUM(fp,WFDIST0(fp,a)*WF0(fp)*(FD0(fp,a))
                **(1+rhocva(a))) ;
 ADVA0(a)       = QVA0(a)/(SUM(f$(FD0(f,a)),deltava(f,a)*FD0(f,a)
                     **(-rhocva(a))))**(-1/rhocva(a));
```

As with the top level the parameters except for the elasticity parameters , rhocva, can be derived from the transaction values. Thus before calibrating the share and shift parameters the initial values for the variables need to be derived. The equations of the initial values of the parameters used are

```
FD0(f,a)                = FACTUSE(f,a) ;
FS0(f)                  = SUM(a,FACTUSE(f,a)) ;
WF0(f)$(FS0(f))         = SUM(a,SAM(f,a))/FS0(f) ;
WFDIST0(f,a)$FD0(f,a)  = (SAM(f,a)/FD0(f,a))/WF0(f) ;
WFDIST0(f,a)$(FD0(f,a) EQ 0) = 0.0 ;
```

Note that factor demands (*FD0*) can be derived from physical quantity data or as value quantities by the assumption that factor prices – *WF0* and *WFDIST0* – are all equal to one. The use of value quantities requires a strong implicit assumption about the properties of each factor type – each factor type is homogenous across all employing activities. This has implications for model applications.

```
* Intermediate Input Demand

 ioqtdqd(c,a)$(QINT0(a)$PQD0(c))
                    = SAM(c,a)/PQD0(c)/QINT0(a) ;
```

Appendix 3 Production System for *smod_t3*

Introduction

The differences in the production systems in *smod_t2* and *smod_t3* requires adding primal and first-order condition equations for the third level nest, adding estimates of the substitution elasticities for the third level, some changes to the sets and set mappings and revisions to the parameter calibration for production. This allows the relaxation of the assumption that the elasticity of substitution between all primary inputs is the same. The most complex part of the process is in the use of the mapping sets in calibration process and the model equations. This requires the use of lefthand side (LHS) and righthand side (RHS) $ controls; such $ control options unlock a lot of power for GAMS code, BUT they are not intuitively obvious.

The three level nesting system developed for *smod_t3* can be extended to produce a generalised nesting system with *n* levels of nested CES functions that can encompass not only primary input substitution possibilities but also substitution possibilities between primary and intermediate inputs required for, *inter alia*, energy and climate change analyses. The extensions can be implemented through appropriate set declarations and minimal changes to the core model equations.

While nested CES functions may initially appear to be a solution to many CGE model issues they are not without costs. Specifically, for each additional nest it is necessary to define substitution elasticities for which there is limited, or no, empirical evidence.

Three Level Production System (*smod_t3*)

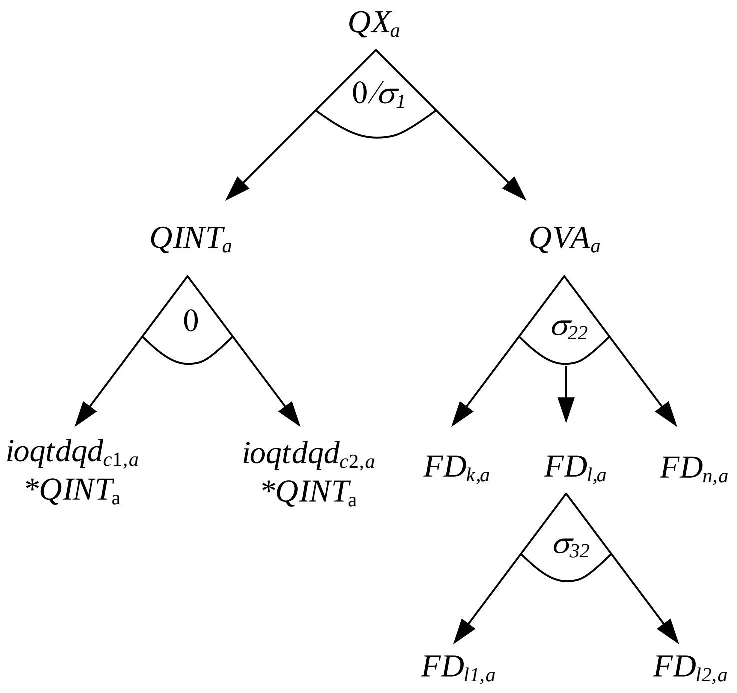
The three level production developed was widely adopted by the early 2000s and was increasingly common by the first half of the 2020s. The method developed here is flexible and is a simplification of the generalised production system used in recent, post 2016, versions of the STAGE and ANARRES models.[23] The generalised system can contain *n*-levels and can be used to include substitution possibilities that include primary and intermediate inputs, e.g., in energy, climate, water, etc., models. The method depends upon the inclusion of appropriate set definitions and mapping sets with minimal need to change the models equations, but it can

23 Prior to the development of this generalised system the STAGE and GLOBE/ANARRES models were 'hard coded' where each level of the nests was represented by primal and first-order conditions. Changes to the nested system required changes, sometimes extensive, to the model equations and calibration systems.

be easy to make mistakes and not so easy to identify the mistakes. For this model the production system is limited to three levels: this is consistent with a pedagogic method involving sequential elaboration.

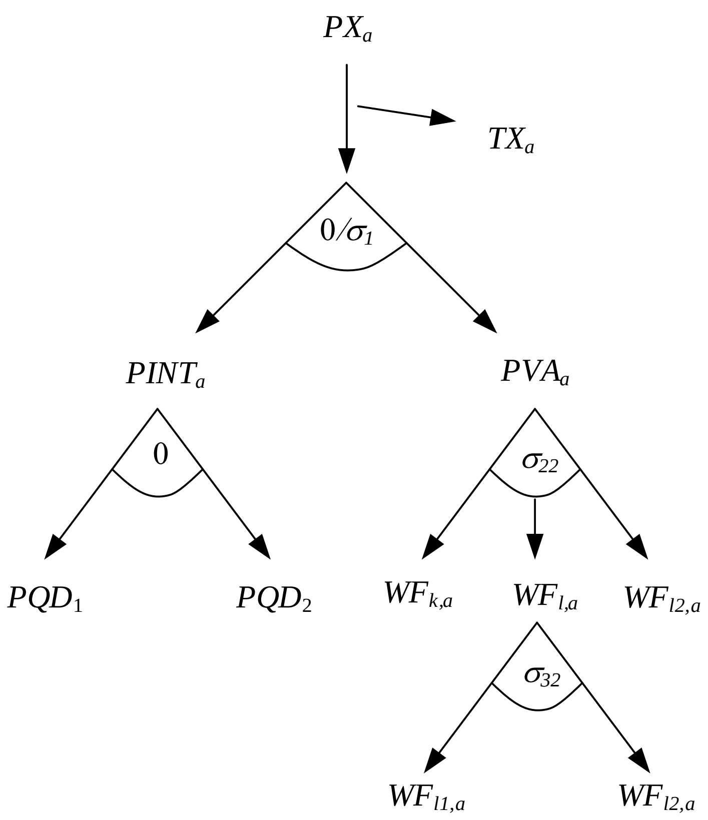
Production relationships by activities are defined by a series of nested Constant Elasticity of Substitution (CES) production functions. In the base version there is a three level production nest, which, in quantity terms, is illustrated in Figure A3.1. For illustration purposes only, two intermediate inputs and five primary inputs ($FD_{k,a}$, $FD_{l1,a}$, $FD_{l2,a}$, $FD_{l3,a}$ and $FD_{n,a}$) together with one aggregate primary input ($FD_{l,a}$) are identified. Activity output is a CES aggregate of the quantities of aggregate intermediate inputs (*QINT*) and value added (*QVA*). Aggregate intermediate inputs are a Leontief aggregate of the (individual) intermediate inputs. Aggregate value added is a CES aggregate of the quantities of two primary and one aggregate inputs demanded by each activity (*FD*). The aggregate primary input is then a CES aggregate of the different primary factors at the third level. The allocation of the finite supplies of factors (*FS*) between competing activities depends upon relative factor prices via first order conditions for optima.

Figure A3.1 Three Level Production System: Quantities



The price relations for the production system are illustrated in Figure A3.2. Note how the prices paid for intermediate inputs (*PQD*) are the same as paid for final demands, i.e., a 'law' of one price relationship holds across all domestic demand. Note also that factor prices are factor and activity specific ($WFA_{f,a}$); in the model $WFA_{f,a} = WF_f * WDIST_{f,a}$, i.e., the labels used in Figure 5.4 are 'abbreviations'

Figure A3.2 Two Level Production System: Prices



Production Block

In this model a three-stage production process is adopted, with the top level as a CES or Leontief function. If a CES is imposed for an activity the value of activity output can be expressed as the volume share weighted sums of the expenditures on inputs after allowing for the production taxes (*TX*), which are assumed to be applied *ad valorem* (P3.2). This requires the definition of aggregate prices for intermediates (*PINT*); these are defined as the intermediate input-output coefficient weighted sum of the prices of intermediate inputs (P3.3), where $ioqtdqd_{c,a}$ are the intermediate input-output coefficients where the output is the aggregate intermediate input (*QINT*).

Production Block Equations: Top Level

$$PX_a = \sum_c ioqxacqx_{a,c} * PXC_c \quad . \qquad \text{(P3.1)}$$

$$PX_a * (1 - TX_a) * QX_a = (PVA_a * QVA_a) + (PINT_a * QINT_a) \quad . \qquad \text{(P3.2)}$$

$$PINT_a = \sum_c \left(ioqtdqd_{c,a} * PQD\right)_c \qquad \text{(P3.3)}$$

$$ADX_a = \left[\left(adxb_a + dabadx_a\right) * ADXADJ\right] + \left(DADX * adx01_a\right) \qquad \text{(P3.4)}$$

$$QX_a = AD_a^x \left(\delta_a^x QVA_a^{-rhoc_a^x} + \left(1 - \delta_a^x\right) QINT_a^{-rhoc_a^x}\right)^{-\frac{1}{rhoc_a^x}} \qquad \forall aqx_a \quad . \qquad \text{(P3.5)}$$

$$\frac{QVA_a}{QINT_a} = \left[\frac{PINT_a}{PVA_a} * \frac{\delta_a^x}{\left(1 - \delta_a^x\right)}\right]^{\frac{1}{\left(1 + rhoc_a^x\right)}} \qquad \forall aqx_a \quad . \qquad \text{(P3.6)}$$

$$QVA_a = ioqvaqx_a * QX_a \qquad \forall aqxn_a \qquad \text{(P3.7a)}$$

$$QINT_a = ioqintqx_a * QX_a \qquad \forall aqxn_a \qquad \text{(P3.7b)}$$

With CES technology the output by an activity, (*QX*) is determined by the aggregate quantities of factors used (*QVA*), i.e., aggregate value added, and aggregate intermediates used (*QINT*), where δ_a^x is the share parameter, $rhoc_a^x$ is the substitution parameter and AD_a^x is the efficiency variable (P3.5). Note how the efficiency/shift factor is defined as a variable and an adjustment mechanism is provided (P3.4), where *adxb* is the base values, *dabadx* is an absolute change in the base value, *ADXADJ* is an equiproportionate (multiplicative) adjustment factor, *DADX* is an additive adjustment factor and *adx*01 is a vector of zeros and non-zeros used to scale the additive adjustment factor. The operation of this type of adjustment equation is explained below for the case of the import duty case. The associated the first order conditions defining the optimum ratios of value added to intermediate inputs can be expressed in terms of the relative prices of value added (*PVA*) and intermediate inputs (*PINT*), see (P3.6).

With Leontief technology at the top level the aggregate quantities of factors used (*QVA*), i.e., aggregate value added, and intermediates used (*QINT*), are determined by simple aggregation functions, (P3.7a) and (P3.7b), where *ioqvaqx* and *ioqintqx* are the (fixed) volume shares of *QVA* and *QINT* (respectively) in *QX*. The choice of top-level aggregation function is controlled by the membership of the set *aqx*, with the membership of *aqxn* being the complement of *aqx*.

Production Block Equations: Second Level

$$ADVA_a = \left[\left(advab_a + dabadva_a\right) * ADVAADJ\right] + \left(DADVA * adva01_a\right) \qquad \text{(X8)}$$

$$QVA_a = AD_a^{va} * \left[\sum_{ff\$\left[map\_va\_f_{ff,a} f \text{ and } \delta_{ff,a}^{va}\right]} \delta_{ff,a}^{va} * ADFD_{ff,a} * FD_{ff,a}^{-\rho_a^{va}}\right]^{-1/\rho_a^{va}} \qquad \forall \rho_a^{va} \qquad \text{(P3.9)}$$

$$\begin{aligned} WF_f * WFDIST_{ff,a} * \left(1 + TF_{ff,a}\right) \\ &= PVA_a * AD_a^{va} * \left[\sum_{ff\$\delta_{ff,a}^{va}} \delta_{ff,a}^{va} * ADFD_{ff,a} * FD_{ff,a}^{-\rho_a^{va}}\right]^{-\left(\frac{1+\rho_a^{va}}{\rho_a^{va}}\right)} * \delta_{ff,a}^{va} * FD_{ff,a}^{\left(-\rho_a^{va}-1\right)} \\ &= PVA_a * QVA_a * AD_a^{va} * \left[\sum_{ff\$\delta_{ff,a}^{va}} \delta_{ff,a}^{va} * ADFD_{ff,a} * FD_{ff,a}^{-\rho_a^{va}}\right]^{-1} \\ &\quad * \delta_{ff,a}^{va} * ADFD_{ff,a}^{-\rho_a^{va}} * \delta_{ff,a}^{va} * FD_{ff,a}^{\left(-\rho_a^{va}-1\right)} \qquad \forall \delta_{f,a}^{va} \text{ and } map\_va\_ff_{ff,a} \end{aligned} \qquad \text{(P3.10)}$$

$$QINTD_c = \sum_a ioqtdqd_{c,a} * QINT_a \,. \qquad \text{(P3.11)}$$

There are two arms to the second level production nest. For aggregate value added (*QVA*) the production function is a multi-factor CES function (P3.9) where δ_a^{va} is the share parameter, $rhoc_a^{va}$ is the substitution parameter and AD_a^{va} is the efficiency factor. The associated first order conditions for profit maximisation (P3.10) determine the wage rate of factors (*WF*), where the ratio of factor payments to factor *f* from activity *a* (*WFDIST*) are included to allow for non-homogenous factors, and is derived directly from the first order condition for profit maximisation as equalities between the wage rates for each factor in each activity and the values of the marginal products of those factors in each activity, Again the efficiency/shift factor is defined as a variable with an adjustment mechanism (P3.8), where *advab* is the base values, *dabadva* is an absolute change in the base value, *ADVAADJ* is an equiproportionate (multiplicative) adjustment factor, *DADVA* is an additive adjustment factor and *adva*01 is a vector of zeros and non-zeros used to scale the additive adjustment factor.

Production Block Equations: Third Level

$$ADFAG_{ff,a} = \left(adfagb_{ff,a} + dabfag_{ff,a}\right) + \left(ADFAGfADJ_{ff} * ADFAGaADJ_{a}\right) \quad \text{(P3.12)}$$

$$FD_{ff,a} = ADFAG_{ff,a} * \left(\sum_{ffp\$\left[\delta^{fd}_{ff,ffp,a} \text{ and } map\_fagg\_f\right]f} \delta^{fd}_{ff,ffp,a} * \left(FD_{ffp,a}\right)^{\rho^{fd}_{ff,a}} \right)^{\left(-1/\rho^{fd}_{ff,a}\right)}$$
$$\forall \sum_{ffp} map\_fagg\_ff_{ff,ffp,a} \quad \text{(P3.13)}$$

$$WFA_{ffp,a} * \left(1 + TF_{ffp,a}\right)$$
$$= WFA_{ff,a} * \left(1 + TF_{ff,a}\right) * FD_{ff,a}$$
$$* \left[\sum_{ffpp\$\left[\delta^{fd}_{ff,ffpp,a} \text{ and } map\_fagg\_ff\right]} \delta^{fd}_{ff,ffpp,a} * FD_{ffpp,a}^{-\rho^{fd}_{ff,a}} \right]^{(-1)} * \delta^{fd}_{ff,ffp,a} * FD_{ffp,a}^{\left(-\rho^{fd}_{ff,a} - 1\right)}$$
$$\forall \delta^{fd}_{ff,ffpp,a}, map\_fagg\_ff_{ff,ffp,a}$$

(P3.14)

The non-factor commodity (intermediate) demands (*QINTD*) are defined as the product of the fixed (Leontief) input coefficients of demand for commodity *c* by activity *a* (*ioqtdqd*), multiplied by the quantity of activity intermediate input (*QINT*) (P3.11).

The primal production functions (P3.13) for the third level define the quantities of natural/individual factors combined to generate the aggregate factor/argument that enters into either the value-added aggregate (P3.9), i.e., members of the set *fagg*. There are efficiency factors ($ADFAG_{ff,a}$), factor shares ($\delta^{fd}_{ff,l,a}$) calibrated from the data and elasticities of substitution, from which the substitution parameters are derived ($\rho^{fd}_{ff,a}$), are exogenously imposed. The matching first order conditions (P3.14) define the wage rate for a specific factor used by a specific activity; these ratios of payments to factor *ff* from activity *a* are included to allow for non-homogenous factors where the differentiation is defined solely in terms of the activity that employs the factor.

The domestic production of commodities (*QXC*) is defined (P3.15) as the sum of the per unit outputs of specific commodities by activities ($ioqxcqx_{c,a}$) scaled by the outputs by each activity (*QX*). This is unchanged from the *smod_t* model. In this model there is a one-to-one relationship between activities and commodities and therefore the matrix *ioqcxqx* is square and block diagonal, i.e., an identity matrix.[24]

Production Block Equations: Commodity Outputs

$$QXC_c = \sum_a ioqxcqx_{c,a} * QX_a \tag{P3.15}$$

Calibration

The equations used to calibrate the top level of *smod_t3* are identical to those used for *smod_t2* and are not repeated here.

The equations for the second level are similar to those used for *smod_t2*, but require the identification of which arguments/inputs are used in the second level. The inputs may be natural or aggregate inputs. In this approach the set of natural (*f*) and aggregate (*fag*) factors is defined as *ff*; this is useful because only natural factors (*f*) actually receive incomes that are parts of factor incomes that are passed down to institutions and therefore all other equations based on natural factors do not need to be changed.

Note that the calibration requires the imposition of exogenous data for the elasticties.

The identification of inputs used in the second level is achieved by setting a 'mapping' set that identifies those members the set *ff* that are used by each activity, *a*: *map_va_ff(ff,a)*. This mapping set can contain members of *ff* that are either natural (*f*) or aggregate (*fagg*) factors. The application of this mapping set can be seen in the calibration code below.

```
rhocva(a)    = (1/ELASTX(a,"sigmava")) - 1 ;
deltava(ff,a)$[map_va_ff(ff,a) AND FD0(ff,a)]
```

24 This formulation is chosen because it leaves 'space' in the model for the subsequent development of a representation of multi-product activities, e.g., as in the STAGE family of models.

```
   = (WFDIST0(ff,a)*WF0(ff)*(FD0(ff,a))**(1+rhocva(a)))
      /SUM(ffp$map_va_ff(ffp,a),WFDIST0(ffp,a)*WF0(ffp)*(FD0(ffp,a))
              **(1+rhocva(a))) ;
ADVA0(a)$SUM(ff$map_va_ff(ff,a),deltava(ff,a)*FD0(ff,a))
      = QVA0(a)/(SUM(ff$map_va_ff(ff,a),deltava(ff,a)*FD0(ff,a)
                          **(-rhocva(a))))**(-1/rhocva(a));
```

When calibrating *deltava* the 'mapping' set defines the elements of *ff* in the numerator by virtue of the $ control on the LHS, while the use of the 'mapping' set in the denominator defines the elements of *ff* that are aggregated. The same logic applies for the shift parameter where the 'mapping' set in the denominator defines the elements of *ff* that are aggregated while the use on the $ control on the LHS simply avoids division by zero. Note this 'mapping' set allows for different elements of *ff* used by each activity.

Checking on the values for *deltava* is critical. The should sum to one for all activities and none can be negative: a 'negative input is an output'. Code is included in the files provided that carryout these checks and cause the programme to abort with error message if the checks fail. Such problems should only be caused by data[25] and mapping set issues.

The third level is concerned with the production of aggregate inputs (*fag*) that will be used, with or without natural factors (*f*), at the second level. Hence the need for a second 'mapping' set that identifies natural factors (*f*) that are used to produce the aggregate factors (*fag*): *map_fagg_ff(ff,ff)*. As before, when calibrating the share parameters, *deltafd*, the 'mapping' set defines the elements of *f* in the numerator, i.e., only natural factors, by virtue of the $ control on the LHS, while the use of the 'mapping' set in the denominator defines the elements of *f* that are aggregated to form the aggregate factors (*fag*). The same logic applies for the shift parameter where the 'mapping' set in the denominator defines the elements of *f* that are aggregated while the use on the $ control on the LHS simply avoids division by zero. Note this 'mapping' set allows for different elements of *f* used by each activity to produce each aggregate.

NOTE: the setup used for calibrating *smod_t3* only allows the production of aggregates (*fag*) at the third level from natural factors (*f*), while at the second level aggregates and natural

25 In SAMs with factor use taxes negative delta can arise if the factor use tax exceeds the payment to the factor, i.e., the more the activity uses of the factor the less its costs of production with no limit.

factors can be used. With appropriate set assignments the third level equations can be used for an *n*-level nest.

The calibration code for the third level is set out below.

```
rhofd(fag,a)    = (1/ELASTF(fag,a)) - 1 ;
deltafd(ff,ffp,a)${map_fagg_ff(ff,ffp,a) AND
                   SUM[ffpp$map_fagg_ff(ff,ffpp,a), FD0(ffpp,a)] }
        = {[WF0(ffp)*WFDIST0(ffp,a)]*(FD0(ffp,a))**[1+rhofd(ff,a)]}
               /{SUM(ffpp$map_fagg_ff(ff,ffpp,a),
                  [WF0(ffpp)*WFDIST0(ffpp,a)]*(FD0(ffpp,a))
                       **[1+rhofd(ff,a)])} ;
adfag(ff,a)${SUM[ffp,map_fagg_ff(ff,ffp,a)] AND
                SUM[ffp$map_fagg_ff(ff,ffp,a),FD0(ffp,a)]   }
          = FD0(ff,a)
              /{ SUM[ffp$map_fagg_ff(ff,ffp,a),
                 deltafd(ff,ffp,a)*FD0(ffp,a)**(-rhofd(ff,a))] }
                    **[-1/rhofd(ff,a)] ;
```

Checking on the values for *deltafd* is critical. They should sum to one for all activities and none can be negative: a ‘negative input is an output’. Code is included in the files provided that carryout these checks and cause the programme to abort with error message if the checks fail. Such problems should only be caused by data[26] and mapping set issues.

Nesting Structure

Also included are codes that report on the nesting structure. These reports are important since the interpretation of model results depends upon the precise nesting structure used, rather than the intended nesting structure.

```
* Reporting the nesting structure
 nest_va(ff,a)$[deltava(ff,a) GT 0.0]     = 1.0 ;
 nest_fd(ff,ffp,a)$[deltafd(ff,ffp,a) GT 0.0]     = 1.0 ;
```

26 In SAMs with factor use taxes negative delta can arise if the factor use tax exceeds the payment to the factor, i.e., the more the activity uses of the factor the less its costs of production with no limit.